Section 3: The equation of a plane

Solutions to Exercise level 2

1. (i) $\underset{\sim}{r}.(3\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k})=(2\underset{\sim}{i}-6\underset{\sim}{j}+\underset{\sim}{k}).(3\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k})=6-6+3$

$\underset{\sim}{r}.(3\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k})=3$

(ii) $3x+y+3z=3$

(iii) Substituting $\underset{\sim}{r}=\underset{\sim}{j}-2\underset{\sim}{k}+\lambda(2\underset{\sim}{i}-5\underset{\sim}{j}+\underset{\sim}{k})$ into the equation of the plane:

$\left[\underset{\sim}{j}-2\underset{\sim}{k}+\lambda(2\underset{\sim}{i}-5\underset{\sim}{j}+\underset{\sim}{k})\right].(3\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k})=3$

$(\underset{\sim}{j}-2\underset{\sim}{k}).(3\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k})+\lambda(2\underset{\sim}{i}-5\underset{\sim}{j}+\underset{\sim}{k}).(3\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k})=3$

$1-6+\lambda(6-5+3)=3$

$4\lambda=8$

$\lambda=2$

Point of intersection is at $\underset{\sim}{r}=\underset{\sim}{j}-2\underset{\sim}{k}+2(2\underset{\sim}{i}-5\underset{\sim}{j}+\underset{\sim}{k})=4\underset{\sim}{i}-9\underset{\sim}{j}$

The point of intersection is (4, -9, 0).

(iv) The direction vector of the line is $\underset{\sim}{d}=2\underset{\sim}{i}-5\underset{\sim}{j}+\underset{\sim}{k}$

The normal vector to the plane is $\underset{\sim}{n}=3\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}$

$\underset{\sim}{n}.\underset{\sim}{d}=\left(3\underset{\sim}{i}+\underset{\sim}{j}+3\underset{\sim}{k}\right).\left(2\underset{\sim}{i}-5\underset{\sim}{j}+\underset{\sim}{k}\right)=6-5+3=4$

$|\underset{\sim}{d}|=\sqrt{2^2+5^2+1^2}=\sqrt{30}$

$|\underset{\sim}{n}|=\sqrt{3^2+1^2+3^2}=\sqrt{19}$

Angle between these vectors is given by $\cos\theta=\dfrac{4}{\sqrt{30}\sqrt{19}}$

$\theta=80.4°$

Angle between the line and the plane $=90°-80.4°=9.6°$

2. (i) Substituting $\underset{\sim}{r}=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\0\end{pmatrix}$ into $\underset{\sim}{r}.\begin{pmatrix}-5\\1\\-7\end{pmatrix}=9$:

$$\left[\begin{pmatrix}2\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\0\end{pmatrix}\right].\begin{pmatrix}-5\\1\\-7\end{pmatrix}=9$$

$$\begin{pmatrix}2\\0\\-1\end{pmatrix}.\begin{pmatrix}-5\\1\\-7\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\0\end{pmatrix}.\begin{pmatrix}-5\\1\\-7\end{pmatrix}=9$$
$$-10+0+7+\lambda(-5+3+0)=9$$
$$-2\lambda=12$$
$$\lambda=-6$$

Position vector of point of intersection is $\underset{\sim}{r}=\begin{pmatrix}2\\0\\-1\end{pmatrix}-6\begin{pmatrix}1\\3\\0\end{pmatrix}=\begin{pmatrix}-4\\-18\\-1\end{pmatrix}$

(ii) The direction vector for the line is $\underset{\sim}{d}=\begin{pmatrix}1\\3\\0\end{pmatrix}$

The normal vector to the plane is $\tilde{n}=\begin{pmatrix}-5\\1\\-7\end{pmatrix}$

$$\underset{\sim}{n}.\underset{\sim}{d}=-5+3+0=-2$$
$$|\underset{\sim}{d}|=\sqrt{1^2+3^2+0^2}=\sqrt{10}$$
$$|\tilde{n}|=\sqrt{5^2+1^2+7^2}=\sqrt{75}$$

Angle between these two vector is given by $\cos\theta=\dfrac{-2}{\sqrt{10}\sqrt{75}}$

$$\theta=94.2^\circ$$

The acute angle between the two vectors is $180^\circ-94.2^\circ=85.8^\circ$

The angle between the line and the plane is $90^\circ-85.8^\circ=4.2^\circ$

3. (i) The vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ lie in the plane.

$$\overrightarrow{AB}=\begin{pmatrix}1\\-1\\1\end{pmatrix}-\begin{pmatrix}3\\0\\2\end{pmatrix}=\begin{pmatrix}-2\\-1\\-1\end{pmatrix}\qquad \overrightarrow{BC}=\begin{pmatrix}2\\3\\-1\end{pmatrix}-\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\4\\-2\end{pmatrix}$$

The point A (3, 0, 2) lies in the plane,
so an equation for the plane is

$$\underset{\sim}{r}=\begin{pmatrix}3\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\1\end{pmatrix}+\mu\begin{pmatrix}1\\4\\-2\end{pmatrix}$$

There are other correct answers – other vectors in the plane could be used for **b** and **c**, and any of the points could be used for **a**.

(ii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

(1) $x = 3 + 2\lambda + \mu$

(2) $y = \lambda + 4\mu$

(3) $z = 2 + \lambda - 2\mu$

(2) – (3) $\Rightarrow y - z = -2 + 6\mu$

$\Rightarrow \mu = \frac{1}{6}(y - z + 2)$

Substituting into (2) $\Rightarrow y = \lambda + \frac{2}{3}(y - z + 2)$

$\Rightarrow 3y = 3\lambda + 2y - 2z + 4$

$\Rightarrow \lambda = \frac{1}{3}(y + 2z - 4)$

Substituting both into (1) $\Rightarrow x = 3 + \frac{2}{3}(y + 2z - 4) + \frac{1}{6}(y - z + 2)$

$\Rightarrow 6x = 18 + 4(y + 2z - 4) + y - z + 2$

$\Rightarrow 6x - 5y - 7z = 4$

This can be written as $\underset{\sim}{r}.\begin{pmatrix} 6 \\ -5 \\ -7 \end{pmatrix} = 4$.

4. If the plane contains the line, then all points on the line satisfy the equation of the plane.
Substituting the equation of the line into the LHS of the equation of the plane:

$\left[3\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})\right].(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$

$= (3\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}).(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}) + \lambda(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}).(3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k})$

$= 9 - 6 - 2 + \lambda(3 - 2 - 1)$

$= 1$

So the equation of the plane is satisfied for all values of λ, and therefore the line lies within the plane.

5. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

(1) $x = 1 + 2\lambda - \mu$

(2) $y = 2 - \lambda + 3\mu$

(3) $z = 3\lambda + 2\mu$

(1) + 2×(2) $\Rightarrow x + 2y = 5 + 5\mu$

$\Rightarrow \mu = \frac{1}{5}(x + 2y - 5)$

Substituting into (1) $\Rightarrow x = 1 + 2\lambda - \frac{1}{5}(x + 2y - 5)$

$\Rightarrow 5x = 5 + 10\lambda - x - 2y + 5$

$\Rightarrow \lambda = \frac{1}{5}(3x + y - 5)$

Substituting both into (3) $\Rightarrow z = \frac{3}{5}(3x + y - 5) + \frac{2}{5}(x + 2y - 5)$

$\Rightarrow 5z = 3(3x + y - 5) + 2(x + 2y - 5)$

$\Rightarrow 11x + 7y - 5z = 25$