Section 3: The equation of a plane

Solutions to Exercise level 1

1. The normal vector is $\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$, so the equation has the form $5x + y - 2z = d$.

   The plane contains the point (2, -3, 1), so $5 \times 2 - 3 - 2 \times 1 = d$

   $10 - 3 - 2 = d$

   $d = 5$

   The equation of the plane is $5x + y - 2z = 5$.

2. The normal vector is $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, so the equation has the form $\underset{\sim}{r}.\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = d$.

   The plane contains the point (1, 4, 0), so substituting the position vector $\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$

   gives $\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}.\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = d$

   $d = 3 - 4 + 0 = -1$

   The equation of the plane is $\underset{\sim}{r}.\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = -1$.

3. (i) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

   $\overrightarrow{AB}.\begin{pmatrix} 11 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.\begin{pmatrix} 11 \\ 6 \\ 1 \end{pmatrix} = 11 - 12 + 1 = 0$

   so $\overrightarrow{AB}$ is perpendicular to $\begin{pmatrix} 11 \\ 6 \\ 1 \end{pmatrix}$.

$$\overrightarrow{BC} = \begin{pmatrix}1\\2\\5\end{pmatrix} - \begin{pmatrix}3\\-1\\1\end{pmatrix} = \begin{pmatrix}-2\\3\\4\end{pmatrix}$$

$$\overrightarrow{BC}.\begin{pmatrix}11\\6\\1\end{pmatrix} = \begin{pmatrix}-2\\3\\4\end{pmatrix}.\begin{pmatrix}11\\6\\1\end{pmatrix} = -22+18+4=0$$

so $\overrightarrow{BC}$ is perpendicular to $\begin{pmatrix}11\\6\\1\end{pmatrix}$.

(ii) The vector $\begin{pmatrix}11\\6\\1\end{pmatrix}$ is normal to the plane, so the equation of the plane is of the form $11x+6y+z=d$

Substituting (2, 1, 0) gives $22+6=d$

$d=28$

The equation of the plane is $11x+6y+z=28$

4. $\overrightarrow{PQ} = \begin{pmatrix}0\\1\\-1\end{pmatrix} - \begin{pmatrix}1\\4\\2\end{pmatrix} = \begin{pmatrix}-1\\-3\\-3\end{pmatrix}$

$\overrightarrow{PR} = \begin{pmatrix}3\\-2\\1\end{pmatrix} - \begin{pmatrix}1\\4\\2\end{pmatrix} = \begin{pmatrix}2\\-6\\-1\end{pmatrix}$

An equation for the plane is $\underset{\sim}{r} = \begin{pmatrix}1\\4\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\3\end{pmatrix} + \mu\begin{pmatrix}2\\-6\\-1\end{pmatrix}$

(Note: other correct forms are possible)

5. (i) (a) A general point on the line is $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+2\lambda\\3\\4-\lambda\end{pmatrix}$

Substituting into $2x-3y+z=3$:

$$2(1+2\lambda)-3(3)+(4-\lambda)=3$$
$$2+4\lambda-9+4-\lambda=3$$
$$3\lambda=6$$
$$\lambda=2$$
$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+2\times2\\3\\4-2\end{pmatrix}=\begin{pmatrix}5\\3\\2\end{pmatrix}$$

So the point of intersection is $(5, 3, 2)$.

(b) The direction vector for the line is $\underset{\sim}{d}=\begin{pmatrix}2\\0\\-1\end{pmatrix}$

The normal vector to the plane is $\tilde{n}=\begin{pmatrix}2\\-3\\1\end{pmatrix}$

$$\underset{\sim}{n}.\underset{\sim}{d}=4+0-1=3$$
$$|\underset{\sim}{d}|=\sqrt{2^2+0^2+1^2}=\sqrt{5}$$
$$|\tilde{n}|=\sqrt{2^2+3^2+1^2}=\sqrt{14}$$

Angle between these two vector is given by $\cos\theta=\dfrac{3}{\sqrt{5}\sqrt{14}}$

$$\theta=69.0^\circ$$

The angle between the line and the plane is $90^\circ-69.0^\circ=21.0^\circ$

(ii) (a) A general point on the line is $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2+\lambda\\-1+3\lambda\\5+4\lambda\end{pmatrix}$

Substituting into $3x+y+4z=3$:

$$3(2+\lambda)+(-1+3\lambda)+4(5+4\lambda)=3$$
$$6+3\lambda-1+3\lambda+20+16\lambda=3$$
$$22\lambda=-22$$
$$\lambda=-1$$
$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2-1\\-1-3\\5-4\end{pmatrix}=\begin{pmatrix}1\\-4\\1\end{pmatrix}$$

So the point of intersection is $(1, -4, 1)$

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(b) The direction vector for the line is $\underset{\sim}{d} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

The normal vector to the plane is $\tilde{n} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$\underset{\sim}{n}.\underset{\sim}{d} = 3 + 3 + 16 = 22$

$|\underset{\sim}{d}| = \sqrt{1^2 + 3^2 + 4^2} = \sqrt{26}$

$|\tilde{n}| = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$

Angle between these two vector is given by $\cos\theta = \dfrac{22}{\sqrt{26}\sqrt{26}}$

$\theta = 32.2^\circ$

The angle between the line and the plane is $90^\circ - 32.2^\circ = 57.8^\circ$

(iii) (a) A general point on the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+4\lambda \\ 3-3\lambda \\ -2+2\lambda \end{pmatrix}$

Substituting into $5x + 2y + 7z = 11$:

$$5(1+4\lambda) + 2(3-3\lambda) + 7(-2+2\lambda) = 11$$

$$5 + 20\lambda + 6 - 6\lambda - 14 + 14\lambda = 11$$

$$28\lambda = 14$$

$$\lambda = \tfrac{1}{2}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+2 \\ 3-\frac{3}{2} \\ -2+1 \end{pmatrix} = \begin{pmatrix} 3 \\ \frac{3}{2} \\ -1 \end{pmatrix}$$

So the point of intersection is $(3, \frac{3}{2}, -1)$.

(b) The direction vector for the line is $\underset{\sim}{d} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$

The normal vector to the plane is $\tilde{n} = \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix}$

$\underset{\sim}{n}.\underset{\sim}{d} = 20 - 6 + 14 = 28$

$|\underset{\sim}{d}| = \sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$

$|\tilde{n}| = \sqrt{5^2 + 2^2 + 7^2} = \sqrt{78}$

Angle between these two vector is given by $\cos\theta = \frac{28}{\sqrt{29}\sqrt{78}}$

$\theta = 53.9°$

The angle between the line and the plane is $90° - 53.9° = 36.1°$