Section 2: The vector equation of a line

Solutions to Exercise level 2

1. (i) Line AB: Direction vector $= \begin{pmatrix} 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Equation of line is $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Line CD: Direction vector $= \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

Equation of line is $\underset{\sim}{r} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

(ii) At point of intersection: $\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

$$2 + \lambda = 6 - 3\mu \quad \Rightarrow \lambda = 4 - 3\mu$$
$$3 - 2\lambda = 5\mu$$
$$3 - 2(4 - 3\mu) = 5\mu$$
$$3 - 8 + 6\mu = 5\mu$$
$$\mu = 5$$

Point of intersection is $\underset{\sim}{r} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + 5\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -9 \\ 25 \end{pmatrix}$

The point of intersection has coordinates (-9. 25)

(iii) Angle between line is angle between direction vectors.

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 5 \end{pmatrix} = (1 \times -3) + (-2 \times 5) = -13$$

$$\left| \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

$$\left| \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}$$

$$\cos\theta = \frac{-13}{\sqrt{5}\sqrt{34}}$$

$$\theta = 175.6° \text{ (1 d.p.)}$$

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2. (i) Line AB: Direction vector $= \begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Equation of line is $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

Line BC: Direction vector $= \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Equation of line is $\underset{\sim}{r} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Line CD: Direction vector $= \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Equation of line is $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Line AD: Direction vector $= \begin{pmatrix} 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Equation of line is $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(ii) AB is parallel to DC, and BC is parallel to AD, ABCD is a parallelogram.

(iii) Angle ABC is angle between vectors $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$.

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} . \begin{pmatrix} -2 \\ -2 \end{pmatrix} = (-1 \times -2) + (3 \times -2) = -4$$

$$\left| \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\left| \begin{pmatrix} -2 \\ -2 \end{pmatrix} \right| = \sqrt{(-2)^2 + (-2)^2} = 2\sqrt{2}$$

$$\cos\theta = \frac{-4}{2\sqrt{10}\sqrt{2}}$$

$$\theta = 116.6^\circ$$

By symmetry, angle CDA is also equal to 116.6°, and angles BCD and DAB are both 180° - 116.6° =63.4°

3. At point of intersection of $\underset{\sim}{r_1} = -6\underset{\sim}{j} + \lambda(\underset{\sim}{i} + \underset{\sim}{j})$ and $\underset{\sim}{r_2} = 9\underset{\sim}{j} + \mu(3\underset{\sim}{i} - 2\underset{\sim}{j})$:

$$-6\underset{\sim}{j} + \lambda(\underset{\sim}{i} + \underset{\sim}{j}) = 9\underset{\sim}{j} + \mu(3\underset{\sim}{i} - 2\underset{\sim}{j})$$

$$\lambda = 3\mu$$

$-6+\lambda = 9-2\mu$

$-6+3\mu = 9-2\mu$

$5\mu = 15$

$\mu = 3$

The point of intersection is $\underset{\sim}{r} = 9\underset{\sim}{j} + 3(3\underset{\sim}{i} - 2\underset{\sim}{j}) = 9\underset{\sim}{i} + 3\underset{\sim}{j}$

To check whether this point lies on the line $\underset{\sim}{r}_3 = -15\underset{\sim}{j} + \delta(\underset{\sim}{i} + 2\underset{\sim}{j})$:

$9\underset{\sim}{i} + 3\underset{\sim}{j} = -15\underset{\sim}{j} + \delta(\underset{\sim}{i} + 2\underset{\sim}{j})$

$9\underset{\sim}{i} + 18\underset{\sim}{j} = \delta(\underset{\sim}{i} + 2\underset{\sim}{j})$

$9(\underset{\sim}{i} + 2\underset{\sim}{j}) = \delta(\underset{\sim}{i} + 2\underset{\sim}{j})$

So the point (9, 3) does lie on this line.
Therefore all three lines pass through the point (9, 3).

4. The angle between the lines is the angle between $\begin{pmatrix}0\\5\\1\end{pmatrix}$ and $\begin{pmatrix}4\\2\\2\end{pmatrix}$

$$\begin{pmatrix}0\\5\\1\end{pmatrix}.\begin{pmatrix}4\\2\\2\end{pmatrix} = (0\times 4)+(5\times 2)+(1\times 2) = 12$$

$$\left|\begin{pmatrix}0\\5\\1\end{pmatrix}\right| = \sqrt{0^2+5^2+1^2} = \sqrt{26} \qquad \left|\begin{pmatrix}4\\2\\2\end{pmatrix}\right| = \sqrt{4^2+2^2+2^2} = \sqrt{24}$$

$$\cos\theta = \frac{12}{\sqrt{26}\sqrt{24}}$$

$$\theta = 61.3^\circ$$

5. At intersection, $\underset{\sim}{k} + \lambda(\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k}) = 2\underset{\sim}{i} + \underset{\sim}{j} + \mu(3\underset{\sim}{j} + 5\underset{\sim}{k})$

Equating coefficients of $\underset{\sim}{i}$: $\lambda = 2$

Equating coefficients of $\underset{\sim}{j}$: $-\lambda = 1+3\mu \Rightarrow -2 = 1+3\mu \Rightarrow \mu = -1$

Equating coefficients of $\underset{\sim}{k}$: $1-3\lambda = 5\mu$

LHS $= 1-3\times 2 = -5$

RHS $= 5\times -1 = -5$

All three equations are satisfied by $\lambda = 2, \mu = -1$, so the lines intersect.

Position vector of point of intersection $= \underset{\sim}{k} + 2(\underset{\sim}{i} - \underset{\sim}{j} - 3\underset{\sim}{k}) = 2\underset{\sim}{i} - 2\underset{\sim}{j} - 5\underset{\sim}{k}$

The point of intersection is (2, -2, -5).

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6. $\underset{\sim}{r} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})$

If point A lies on the line, $\underset{\sim}{i} - 2\underset{\sim}{j} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})$

$$-\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} = \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})$$

This is not true for any value of λ, so the point A does not lie on the line.

If point B lies on the line, $3\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})$

$$\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k} = \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})$$

This is true for $\lambda = 1$, so the point B lies on the line.

If point C lies on the line, $\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})$

$$-\underset{\sim}{i} + 4\underset{\sim}{j} = +\lambda(\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})$$

This is not true for any value of λ, so the point C does not lie on the line.

7. (i) Direction vector $= 7\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k} - (4\underset{\sim}{i} - 8\underset{\sim}{j})$

$$= 3\underset{\sim}{i} + 6\underset{\sim}{j} + 3\underset{\sim}{k}$$

$$= 3(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$$

Vector equation of line AB is $\underset{\sim}{r} = 4\underset{\sim}{i} - 8\underset{\sim}{j} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$

(ii) At point of intersection, $4\underset{\sim}{i} - 8\underset{\sim}{j} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}) = 8\underset{\sim}{i} - 2\underset{\sim}{j} + \mu(\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$

Equating coefficients of $\underset{\sim}{i}$: $4 + \lambda = 8 + \mu$ (1)

Equating coefficients of $\underset{\sim}{j}$: $-8 + 2\lambda = -2 + \mu$ (2)

Equating coefficients of $\underset{\sim}{k}$: $\lambda = -\mu$ (3)

Substituting (3) into (1): $4 - \mu = 8 + \mu \Rightarrow -4 = 2\mu \Rightarrow \mu = -2$

Substituting (3) into (2): $-8 - 2\mu = -2 + \mu \Rightarrow -6 = 3\mu \Rightarrow \mu = -2$

All 3 equations are satisfied by $\mu = -2, \lambda = 2$, so the lines intersect.

Position vector of point of intersection is $\underset{\sim}{r} = 4\underset{\sim}{i} - 8\underset{\sim}{j} + 2(\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$

$$= 6\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$$

Point of intersection is (6, -4, 2)

8. $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 3 \\ k \end{pmatrix}$

(1) $2 + \lambda = 3 + \mu$

(2) $3 - 2\lambda = -4 + 3\mu$

(3) $1 - 3\lambda = 2 + k\mu$

$2 \times (1) + (2) \quad \Rightarrow 7 = 2 + 5\mu \quad \Rightarrow \mu = 1, \lambda = 2$

Substituting into (3) gives $1 - 6 = 2 + k \quad \Rightarrow k = -7$

Position vector of P is given by $r = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$

so P = (4, -1, -5)