Section 2: Proof by Induction

Solutions to Exercise level 3

1. $2(2)+3(2^2)+4(2^3)+\dots+(n+1)(2^n)=\sum_{r=1}^{n}(r+1)2^r$

To prove $\sum_{r=1}^{n}(r+1)2^r = n(2^{n+1})$

Step 1: When $n=1$ L.H.S. $=2\times 2=4$

R.H.S. $=1\times 2^2=4$

so it is true for $n=1$.

Step 2: Assume $\sum_{r=1}^{k}(r+1)2^r = k(2^{k+1})$

$$\begin{aligned}\sum_{r=1}^{k+1}(r+1)2^r &= k(2^{k+1})+(k+2)2^{k+1}\\ &= k(2^{k+1})+k(2^{k+1})+2\times 2^{k+1}\\ &= 2k(2^{k+1})+2\times 2^{k+1}\\ &= k(2^{k+2})+2^{k+2}\\ &= (k+1)2^{k+2}\end{aligned}$$

Step 3: So if the result is true for $n=k$, then it is true for $n=k+1$. Since it is true for $n=1$, then it is true for all positive integers greater than or equal to 1 by induction.

We have $2(2)+3(2^2)+4(2^3)+\dots+(n+1)(2^n)$

We want to find $1(2)+2(2^2)+3(2^3)+\dots+98(2^{98})$

$$\begin{array}{cccccccc} & 1(2) & +2(2^2) & +3(2^3) & +\cdots & +98(2^{98}) & S_1\\ + & (2) & +(2^2) & +(2^3) & +\cdots & +(2^{98}) & S_2\\ \hline & 2(2) & +3(2^2) & +4(2^3) & +\cdots & +99(2^{98}) & S_3\end{array}$$

S_2 is a geometric series with first term 2, common ratio 2 and 98 terms.

$$S_2=\frac{2(2^{98}-1)}{2-1}=2(2^{98}-1)$$

S_3 is the previous sum, with $n=98$.

$$S_3=98(2^{99})$$

So $S_1 = S_3 - S_2$

$= 98(2^{99}) - 2(2^{98} - 1)$

$= 98(2^{99}) - 2^{99} + 2$

$= 97(2^{99}) + 2$

2. (i) $F_3, F_4, \ldots F_{10} = 2, 3, 5, 8, 13, 21, 34, 55$

(ii)
$$\begin{aligned}F_{n+5} &= F_{n+4} + F_{n+3}\\ &= F_{n+3} + F_{n+2} + F_{n+3}\\ &= 2F_{n+3} + F_{n+2}\\ &= 2(F_{n+2} + F_{n+1}) + F_{n+2}\\ &= 3F_{n+2} + 2F_{n+1}\\ &= 3(F_{n+1} + F_n) + 2F_{n+1}\\ &= 5F_{n+1} + 3F_n\end{aligned}$$

(iii) To prove that F_{5n} is a multiple of 5 for $n \geq 1$

Step 1: For $n = 1$, $F_5 = 5$
So it is true for $n = 1$

Step 2: Assume F_{5k} is a multiple of 5,
so $F_{5k} = 5p$ for some integer p
Consider $n = k+1$
$F_{5(k+1)} = F_{5k+5}$
Using result from (ii):
$$\begin{aligned}F_{5k+5} &= 5F_{5k+1} + 3F_{5k}\\ &= 5F_{5k+1} + 3 \times 5p\\ &= 5(F_{5k+1} + 3p)\end{aligned}$$
so $F_{5(k+1)}$ is a multiple of 5.

Step 3: So if the result is true for $n = k$, then it is true for $n = k+1$.
Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.