Section 2: Proof by induction

Solutions to Exercise level 2

1. To prove that $\sum_{r=1}^{n} r^2(r+1) = \frac{n(n+1)(n+2)(3n+1)}{12}$.

Step 1: When $n = 1$, L.H.S. $= 1^2 \times 2 = 2$

R.H.S. $= \frac{1 \times 2 \times 3 \times 4}{12} = 2$

So the result is true for $n = 1$.

Step 2: Assume $\sum_{r=1}^{k} r^2(r+1) = \frac{k(k+1)(k+2)(3k+1)}{12}$

$$\sum_{r=1}^{k+1} r^2(r+1) = \frac{k(k+1)(k+2)(3k+1)}{12} + (k+1)^2(k+2)$$

$$= \frac{k(k+1)(k+2)(3k+1) + 12(k+1)^2(k+2)}{12}$$

$$= \frac{(k+1)(k+2)(3k^2 + k + 12k + 12)}{12}$$

$$= \frac{(k+1)(k+2)(3k^2 + 13k + 12)}{12}$$

$$= \frac{(k+1)(k+2)(k+3)(3k+4)}{12}$$

$$= \frac{(k+1)((k+1)+1)((k+1)+2)(3(k+1)+1)}{12}$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

2. To prove that $\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{(n+2)}{2^n}$.

Step 1: When $n = 1$, L.H.S. $= \frac{1}{2}$

R.H.S. $= 2 - \frac{3}{2} = \frac{1}{2}$

So the result is true for $n = 1$.

Step 2: Assume $\sum_{r=1}^{k} \frac{r}{2^r} = 2 - \frac{(k+2)}{2^k}$

$$\sum_{r=1}^{k+1}\frac{r}{2^r} = 2-\frac{(k+2)}{2^k}+\frac{k+1}{2^{k+1}}$$
$$= 2-\frac{2(k+2)-(k+1)}{2^{k+1}}$$
$$= 2-\frac{2k+4-k-1}{2^{k+1}}$$
$$= 2-\frac{(k+1)+2}{2^{k+1}}$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

3. To prove $\sum_{r=1}^{n} 2\times 3^r = 3\left(3^n - 1\right)$

Step 1: When $n = 1$, L.H.S. $= 2\times 3 = 6$

R.H.S. $= 3(3^1 - 1) = 3\times 2 = 6$

So the result is true for $n = 1$.

Step 2: Assume $\sum_{r=1}^{k} 2\times 3^r = 3\left(3^k - 1\right)$

$$\sum_{r=1}^{n} 2\times 3^r = 3\left(3^k - 1\right) + 2\times 3^{k+1}$$
$$= 3\times 3^k - 3 + 2\times 3^{k+1}$$
$$= 3^{k+1} - 3 + 2\times 3^{k+1}$$
$$= 3\times 3^{k+1} - 3$$
$$= 3(3^{k+1} - 1)$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

4. To prove that $\sum_{r=1}^{n} r(r+2) = \frac{n(n+1)(2n+7)}{6}$.

Step 1: When $n = 1$, L.H.S. $= 1\times 3 = 3$

R.H.S. $= \frac{1\times 2\times 9}{6} = 3$

So the result is true for $n = 1$.

Step 2: Assume $\sum_{r=1}^{k} r(r+2) = \frac{k(k+1)(2k+7)}{6}$

$$\sum_{r=1}^{k+1} r(r+2) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$
$$= \frac{k(k+1)(2k+7)+6(k+1)(k+3)}{6}$$
$$= \frac{(k+1)(2k^2+7k++6k+18)}{6}$$
$$= \frac{(k+1)(2k^2+13k+18)}{6}$$
$$= \frac{(k+1)(k+2)(2k+9)}{6}$$
$$= \frac{(k+1)((k+1)+1)(2(k+1)+7)}{6}$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

5. To prove that $\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{(2n+1)}$.

Step 1: When $n = 1$, L.H.S. $= \frac{1}{1\times 3} = \frac{1}{3}$

R.H.S. $= \frac{1}{3}$

So the result is true for $n = 1$.

Step 2: Assume $\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{(2k+1)}$

$$\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)}$$
$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$
$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$
$$= \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$$
$$= \frac{k+1}{2(k+1)+1}$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

6. To prove that $\sum_{r=1}^{n} 2^{r-1} = 2^n - 1$.

Step 1: When $n = 1$, L.H.S. $= 2^0 = 1$
R.H.S. $= 2^1 - 1 = 1$
So the result is true for $n = 1$.

Step 3: Assume $\sum_{r=1}^{k} 2^{r-1} = 2^k - 1$

$$\sum_{r=1}^{k+1} 2^{r-1} = 2^k - 1 + 2^{(k+1)-1}$$
$$= 2^k - 1 + 2^k$$
$$= 2 \times 2^k - 1$$
$$= 2^{k+1} - 1$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

7. To prove that for $u_{n+1} = 3u_n + 2$ and $u_1 = 1$, for $n \geq 1$, $u_n = 2\left(3^{n-1}\right) - 1$

Step 1: When $n = 1$, $u_1 = 2\left(3^0\right) - 1 = 2 \times 1 - 1 = 1$
So the result is true for $n = 1$.

Step 2: Assume $u_k = 2\left(3^{k-1}\right) - 1$

$$u_{k+1} = 3u_k + 2$$
$$= 3\left(2\left(3^{k-1}\right) - 1\right) + 2$$
$$= 3 \times 2 \times 3^{k-1} - 3 + 2$$
$$= 2 \times 3^k - 1$$
$$= 2(3^{(k+1)-1}) - 1$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

8. To prove that for $u_{n+1} = 2u_n + 1$ and $u_1 = 5$, where n is a positive integer, $u_n = 3 \times 2^n - 1$.

Step 1: When $n = 1$, $u_1 = 3 \times 2^1 - 1 = 5$

So the result is true for $n = 1$.

Step 2: Assume $u_k = 3\times 2^k - 1$

$$u_{k+1} = 2u_k + 1$$
$$= 2(3\times 2^k - 1) + 1$$
$$= 3\times 2^{k+1} - 2 + 1$$
$$= 3\times 2^{k+1} - 1$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

9. To prove that if $A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$, $A^n = \begin{pmatrix} 2n+1 & -n \\ 4n & 1-2n \end{pmatrix}$ where n is a positive integer.

Step 1: When $n = 1$, $A^1 = \begin{pmatrix} 2\times 1+1 & -1 \\ 4\times 1 & 1-2\times 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix} = A$

So the result is true for $n = 1$.

Step 2: Assume $A^k = \begin{pmatrix} 2k+1 & -k \\ 4k & 1-2k \end{pmatrix}$

$$A^{k+1} = \begin{pmatrix} 2k+1 & -k \\ 4k & 1-2k \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 3(2k+1)-4k & -(2k+1)+k \\ 12k+4(1-2k) & -4k-(1-2k) \end{pmatrix}$$
$$= \begin{pmatrix} 2k+3 & -k-1 \\ 4k+4 & -2k-1 \end{pmatrix}$$
$$= \begin{pmatrix} 2(k+1)+1 & -(k+1) \\ 4(k+1) & 1-2(k+1) \end{pmatrix}$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

10. To prove that if $M = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$, $M^n = 5^{n-1}\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ where $n \geq 1$

Step 1: For $n = 1$, $M^1 = 5^0\begin{pmatrix}2 & 6\\1 & 3\end{pmatrix} = \begin{pmatrix}2 & 6\\1 & 3\end{pmatrix}$

So the result is true for $n = 1$.

Step 2: Assume $M^k = 5^{k-1}\begin{pmatrix}2 & 6\\1 & 3\end{pmatrix}$

$$\begin{aligned}M^{k+1} &= 5^{k-1}\begin{pmatrix}2 & 6\\1 & 3\end{pmatrix}\begin{pmatrix}2 & 6\\1 & 3\end{pmatrix}\\ &= 5^{k-1}\begin{pmatrix}10 & 30\\5 & 15\end{pmatrix}\\ &= 5^{k-1}\times 5\begin{pmatrix}2 & 6\\1 & 3\end{pmatrix}\\ &= 5^{k}\begin{pmatrix}2 & 6\\1 & 3\end{pmatrix}\end{aligned}$$

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

11. To prove that $n^3 + 3n^2 - 10n$ is a multiple of 3.

Step 1: When $n = 1$, $n^3 + 3n^2 - 10n = 1 + 3 - 10 = -6$ which is a multiple of 3.

So the result is true for $n = 1$.

Step 2: Assume $f(k) = k^3 + 3k^2 - 10k$ is a multiple of 3.

$$\begin{aligned}f(k+1) &= (k+1)^3 + 3(k+1)^2 - 10(k+1)\\ &= k^3 + 3k^2 + 3k + 1 + 3k^2 + 6k + 3 - 10k - 10\\ &= k^3 + 6k^2 - k - 6\\ &= \left(f(k) - 3k^2 + 10k\right) + 6k^2 - k - 6\\ &= f(k) + 3k^2 + 9k - 6\\ &= f(k) + 3(k^2 + 3k - 2)\end{aligned}$$

Since $3(k^2 + 3k - 2)$ is a multiple of 3, then if f(k) is a multiple of 3, then f(k+1) is a multiple of 3.

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.

12. To prove that $3^{2n}-1$ is a multiple of 8.

Step 1: When $n = 1$, $3^{2n}-1=3^2-1=8$ which is a multiple of 8.
So the result is true for $n = 1$.

Step 2: Assume $f(k)=3^{2k}-1$ is a multiple of 8.

$$
\begin{aligned}
f(k+1) &= 3^{2(k+1)}-1 \\
&= 3^{2k}3^2-1 \\
&= 9\times 3^{2k}-1 \\
&= 9\left(f(k)+1\right)-1 \\
&= 9f(k)+8
\end{aligned}
$$

So if f(k) is a multiple of 8, then f(k+1) is a multiple of 8.

Step 3: So if the result is true for $n = k$, then it is true for $n = k + 1$.
Since it is true for $n = 1$, then it is true for all positive integers greater than or equal to 1 by induction.