Section 3: Matrices and simultaneous equations

Solutions to Exercise level 3

1. (i) $\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix}$

For the equations to have a unique solution

$$\det\begin{pmatrix} a & b \\ b & a \end{pmatrix} \neq 0$$

$$a^2 - b^2 \neq 0$$

$$(a+b)(a-b) \neq 0$$

$$a \neq \pm b$$

(ii) $\det\begin{pmatrix} 2 & b \\ b & 2 \end{pmatrix} = 4 - b^2$

$$\begin{pmatrix} 2 & b \\ b & 2 \end{pmatrix}^{-1} = \frac{1}{4-b^2}\begin{pmatrix} 2 & -b \\ -b & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & b \\ b & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ b \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4-b^2}\begin{pmatrix} 2 & -b \\ -b & 2 \end{pmatrix}\begin{pmatrix} 1 \\ b \end{pmatrix} = \frac{1}{4-b^2}\begin{pmatrix} 2-b^2 \\ b \end{pmatrix}$$

so $x = \dfrac{2-b^2}{4-b^2}$, $y = \dfrac{b}{4-b^2}$

(ii) If the solution lies on the line $y = x$,

$$\frac{2-b^2}{4-b^2} = \frac{b}{4-b^2}$$

$$\Rightarrow 2 - b^2 = b$$

$$\Rightarrow b^2 + b - 2 = 0$$

$$\Rightarrow (b+2)(b-1) = 0$$

$$\Rightarrow b = -2 \text{ or } b = 1$$

But since $a = 2$, $b \neq \pm 2$

therefore $b = 1$.

2. (i) $\mathbf{T}\mathbf{T}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} \lambda & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & \lambda \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \lambda+1 & 0 & 0 \\ \lambda-1 & 2 & \lambda-1 \\ 0 & 0 & 1+\lambda \end{pmatrix}$

For $\lambda = 1$, this gives the identity matrix, so $\lambda = 1$.

(ii) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} Z \\ Y \\ X \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Using the result from (i), the inverse of T is $\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

so $\begin{pmatrix} Z \\ Y \\ X \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+b-c \\ a-b+c \\ -a+b+c \end{pmatrix}$

so $Z = \frac{1}{2}(a+b-c)$

$Y = \frac{1}{2}(a-b+c)$

$X = \frac{1}{2}(-a+b+c)$

(iii) $xy = Z, xz = Y, yz = X$

so $\frac{YZ}{X} = \frac{xyxz}{yz} = x^2$

$\Rightarrow x^2 = \frac{\frac{1}{2}(a-b+c)\frac{1}{2}(-a+b+c)}{\frac{1}{2}(a+b-c)} = \frac{(a-b+c)(a+b-c)}{2(-a+b+c)}$

$\Rightarrow x = \pm\sqrt{\frac{(a-b+c)(a+b-c)}{2(-a+b+c)}}$

Similarly $y = \pm\sqrt{\frac{(a+b-c)(-a+b+c)}{2(a-b+c)}}$

and $z = \pm\sqrt{\frac{(a-b+c)(-a+b+c)}{2(a+b-c)}}$

3. (i) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

$\det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \cos^2\theta + \sin^2\theta = 1$

so the determinant is not zero for any value of θ, and so the simultaneous equations have a unique solution for all values of θ.

(ii) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3\cos\theta + 4\sin\theta \\ -3\sin\theta + 4\cos\theta \end{pmatrix}$

For $\theta = 30°$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3\cos 30° + 4\sin 30° \\ -3\sin 30° + 4\cos 30° \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\sqrt{3} + 2 \\ -\frac{3}{2} + 2\sqrt{3} \end{pmatrix}$

so the solution is $x = \frac{3}{2}\sqrt{3} + 2,\ y = -\frac{3}{2} + 2\sqrt{3}$

(iii) If a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ satisfies the simultaneous equations, then $\begin{pmatrix} x \\ y \end{pmatrix}$ is mapped onto $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ by an anticlockwise rotation through θ about the origin. Since a rotation about the origin does not affect the modulus of the vector, all such vectors will have modulus 5 (as $\sqrt{3^2 + 4^2} = 5$). This describes a circle with radius 5 and centre at the origin, so the locus of points is given by

$$x^2 + y^2 = 5$$

Alternatively:

$x = 3\cos\theta + 4\sin\theta,\ y = -3\sin\theta + 4\cos\theta$

Squaring: $x^2 = 9\cos^2\theta + 24\cos\theta\sin\theta + 16\sin^2\theta$

$y^2 = 9\sin^2\theta - 24\cos\theta\sin\theta + 16\cos^2\theta$

Adding: $x^2 + y^2 = 9\left(\cos^2\theta + \sin^2\theta\right) + 16\left(\sin^2\theta + \cos^2\theta\right)$

$\Rightarrow x^2 + y^2 = 9 + 16$

$\Rightarrow x^2 + y^2 = 25$

4. (i) $\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & a \\ 1 & 0 & b \end{vmatrix} = 1(b - 0) + 1(a - 0) = a + b$

Matrix of cofactors $= \begin{pmatrix} b & a & -1 \\ -b & b & 1 \\ a & -a & 1 \end{pmatrix}$

$$A^{-1} = \frac{1}{a+b}\begin{pmatrix} b & -b & a \\ a & b & -a \\ -1 & 1 & 1 \end{pmatrix}$$

(ii) There is a unique solution provided that $a + b \neq 0$.

(iii) If there are solutions but not a unique solution, then $a + b = 0$, and the equations must be consistent.

The equations are:

(1) $x + y = c$

(2) $y + az = 1$

(3) $x + bz = 1$

(2) + (3) gives $x + y + (a+b)z = 2$

Since $a + b = 0$, this gives $x + y = 2$

So for consistency with (1), $c = 2$.

So $a + b = 0$ and $c = 2$.

5. (i) If the planes meet at a point, the determinant of the matrix is non-zero.

$$\begin{vmatrix} 2 & a & b \\ 4 & c & d \\ 8 & 4 & 12 \end{vmatrix} = 2(12c - 4d) + a(8d - 48) + b(16 - 8c)$$

$$= 24c - 8d + 8ad - 48a + 16b - 8bc$$

$$= 8(-6a + 2b + 3c - d + ad - bc)$$

So any values for which the determinant is not zero can be used, e.g. $a = 1$ and $b = c = d = 0$, and p, q can take any values.

(ii) In this case, the determinant is zero, none of the planes are parallel so no row is a multiple of any other, and there is no solution to the equations. Choosing values that make the determinant zero but do not make any row a multiple of any other:

e.g. $a = b = d = 1$, $c = 2$

This gives equations (1) $2x + y + z = p$

(2) $4x + 2y + z = q$

(3) $8x + 4y + 12z = 20$

2(1)-(2) gives $z = 2p - q$

(3)-2(2) gives $10z = 20 - 2q \quad \Rightarrow 5z = 10 - q$

If these are to be consistent, $5(2p - q) = 10 - q$

$10p - 5q - = 10 - q$

$10p - 6q = 10$

So for the equations to be inconsistent, choose any values of p and q for which this is not true, e.g. $p = 0$, $q = 1$.

(iii) If two of the planes are parallel, then for example $c = 2$, $d = 6$ means the second and third equations are parallel. The values of a and b can be any for which the first line is not parallel, e.g. $a = b = 0$. In this case the value of q can be anything other than 10 (so that the second equation is not the same as the first) and p can take any value.

(iv) If all planes are parallel, then each row must be a multiple of the others. So $a = 1$, $b = 3$, $c = 2$, $d = 6$.

If all three planes are distinct, then p and q must be such that the three equations are all different. e.g. $p = q = 20$

(v) This is similar to (ii) but the equations must be consistent.
So use the same values for a, b, c and d
e.g. $a = b = d = 1, c = 2$
but this time choose values for p and q that satisfy $10p - 6q = 10$
e.g. $p = 1, q = 0$.