Section 3: Invariance

Solutions to Exercise level 3

1. $\begin{pmatrix} X \\ Y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 2y-3 \\ x+1 \\ 1 \end{pmatrix}$

For invariant points, $\begin{pmatrix} 2y-3 \\ x+1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -x+2y-3 \\ x-y+1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Adding: $y-2=0$

$y=2, x=1$

The point (1, 2) is an invariant point.

For invariant lines, $\begin{pmatrix} X \\ mX+c \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & -3 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx+c \\ 1 \end{pmatrix} = \begin{pmatrix} 2mx+2c-3 \\ x+1 \\ 1 \end{pmatrix}$

Substituting $X = 2mx+2x-3$ into $mX+c = x+1$

gives $m(2mx+2c-3)+c = x+1$

$2m^2x+2mc-3m+c = x+1$

Equating coefficients of x gives $2m^2 = 1$

$$m = \pm\frac{1}{\sqrt{2}}$$

Equating constant terms gives $2mc-3m+c=1$

$$c = \frac{1+3m}{2m+1} = \frac{1\pm\frac{3}{\sqrt{2}}}{1\pm\frac{2}{\sqrt{2}}} = \frac{\sqrt{2}\pm 3}{\sqrt{2}\pm 2}$$

$$c = 2-\tfrac{1}{2}\sqrt{2} \text{ or } 2+\tfrac{1}{2}\sqrt{2}$$

So the invariant lines are $y = \frac{1}{\sqrt{2}}x+2-\frac{1}{2}\sqrt{2}$ and $y = -\frac{1}{\sqrt{2}}x-2+\frac{1}{2}\sqrt{2}$

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2. $$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 4 & -3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 2x-y+3 \\ 3x+4y-3 \\ 1 \end{pmatrix}$$

For invariant points, $\begin{pmatrix} 2x-y+3 \\ 3x+4y-3 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x-y+3 \\ 3x+3y-3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x-y+3=0$

$x+y-1=0$

Adding: $2x+2=0$

$x=-1,\ y=2$

The point (-1, 2) is an invariant point.

For invariant lines,

$$\begin{pmatrix} x' \\ mx'+c \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 3 \\ 3 & 4 & -3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ mx+c \\ 1 \end{pmatrix} = \begin{pmatrix} 2x-mx-c+3 \\ 3x+4mx+4c-3 \\ 1 \end{pmatrix}$$

Substituting $x'=2x-mx-c+3$ into $mx'+c=3x+4mx+4c-3$

gives $m(2x-mx-c+3)+c=3x+4mx+4c-3$

$2mx-m^2x-mc+3m+c=3x+4mx+4c-3$

Equating coefficients of x gives $2m-m^2=3+4m$

$m^2+2m+3=0$

This quadratic has no real roots, so there are no invariant lines of the form $y=mx+c$. The only other possibility is a line of the form $x=c$, but since x maps to a function of y then the image of x cannot be constant. So there are no invariant lines.