Section 3: Invariance

Solutions to Exercise level 2

1. At invariant points: $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} 4x+3y \\ -3x-2y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 3x+3y \\ -3x-3y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The invariant points satisfy the equation $x + y = 0$
so $y = -x$ is a line of invariant points.

For points on invariant lines $y = mx + c$,
image points are given by $\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} 4x+3mx+3c \\ -3x-2mx-2c \end{pmatrix}$
The image point must lie on the line $y = mx + c$,
so $-3x - 2mx - 2c = m(4x + 3mx + 3c) + c$

$$(3m^2 + 6m + 3)x + 3mc + 3c = 0$$

$$3(m^2 + 2m + 1)x + 3c(m + 1) = 0$$

$$(m + 1)^2 x + c = 0$$

This must be true for all values of x, so $m = -1$ and c can take any value
So the invariant lines are $y = -x + c$
(note that the case $c = 0$) is the line of invariant points already found)

2. (i) $M^2 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4}+\frac{1}{4} & \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{4} & \frac{1}{4}+\frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) For invariant points, $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \frac{\sqrt{3}}{2}x + \frac{1}{2}y \\ \frac{1}{2}x - \frac{\sqrt{3}}{2}y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = x \quad \Rightarrow x\sqrt{3} + y = 2x \quad \Rightarrow y = x(2-\sqrt{3})$$

$$\frac{1}{2}x - \frac{\sqrt{3}}{2}y = y \quad \Rightarrow x - \sqrt{3}y = 2y \quad \Rightarrow y(2+\sqrt{3}) = x$$

$$\Rightarrow y = \frac{x}{(2+\sqrt{3})}$$

$$\Rightarrow y = x(2-\sqrt{3})$$

So the equation of the line of invariant points is $y = x(2-\sqrt{3})$.

(iii) The transformation is a reflection in the line $y = x(2-\sqrt{3})$.
The result from part (i) shows that M is its own inverse, which is true of a reflection – performing a reflection twice returns to the original object.

(iv) The other invariant lines for a reflection are perpendicular to the mirror line, so they are of the form $y = -\dfrac{x}{2-\sqrt{3}} + c$, or $y = -(2+\sqrt{3})x + c$

3. (i) At invariant points, $\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{pmatrix} -x+y \\ -4x+3y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -2x+y \\ -4x+2y \end{pmatrix} = 0$$

The invariant points satisfy the equation $-2x + y = 0$
so the line of invariant points is $y = 2x$.

(ii) For points on invariant lines $y = mx + c$,
image points are given by $\begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} -x+mx+c \\ -4x+3mx+3c \end{pmatrix}$
The image point must lie on the line $y = mx + c$,
so $-4x + 3mx + 3c = m(-x + mx + c) + c$

$$-4x + 3mx + 3c = -mx + m^2x + mc + c$$

$$(m^2 - 4m + 4)x + mc - 2x = 0$$

$$(m-2)^2 x + c(m-2) = 0$$

This must be true for all values of x, so $m = 2$, and c can take any value.
So the invariant lines have the form $y = 2x + c$.

The invariant lines are parallel to the line of invariant points (the shear line). This is what you would expect since in a shear points are translated parallel to the shear line.