Section 3: Matrices and simultaneous equations

Solutions to Exercise level 2

1. (i) $\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$\det\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix} = 5$

Inverse of $\begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$ is $\frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 15 \\ -10 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

There is a unique solution: $x = 3$, $y = -2$.

(ii) $\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

$\det\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} = 0$ so there is not a unique solution.

The equations are multiples of each other so they are consistent, and there are infinitely many solutions.

2. $\begin{pmatrix} 1 & 3 & 1 \\ 2 & -1 & -4 \\ 4 & 5 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 9 \end{pmatrix}$

Inverse matrix $= \frac{1}{7}\begin{pmatrix} 23 & 14 & -11 \\ -10 & -7 & 6 \\ 14 & 7 & -7 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 23 & 14 & -11 \\ -10 & -7 & 6 \\ 14 & 7 & -7 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \\ 9 \end{pmatrix}$

$= \frac{1}{7}\begin{pmatrix} 35 \\ -7 \\ 14 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$

The solution of the equations is $x = 5$, $y = -1$ and $z = 2$.

3. (i) $\det A = k(k-12) - 4(0-8)$

$= k^2 - 12k + 32$

When $k = 2$, $\det A = 4 - 24 + 32 = 12$

$\det A \neq 0$, so A is non-singular when $k = 2$.

(ii) When $k = 4$, $\det A = 16 - 48 + 32 = 0$

so the equations do not have a unique solution.

$$4x + 4y \quad = 6 \quad (1)$$
$$4y + 4z = 8 \quad (2)$$
$$2x + 3y + z = 1 \quad (3)$$

(1)-2(3) $\Rightarrow -2y - 2z = 4 \quad \Rightarrow y + z = -2$

(2) $\Rightarrow y + z = 2$

The equations are inconsistent, so there are no solutions.

4. (i) $$AB = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & a \end{pmatrix}\begin{pmatrix} 3a-1 & a+1 & -4 \\ 1 & 2a-1 & -2 \\ -3 & -3 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 6a-2-1-3 & 2a+2-2a+1-3 & -8+2+6 \\ 0+3-3 & 0+6a-3-3 & 0-6+6 \\ 3a-1+1-3a & a+1+2a-1-3a & -4-2+6a \end{pmatrix}$$

$$= \begin{pmatrix} 6a-6 & 0 & 0 \\ 0 & 6a-6 & 0 \\ 0 & 0 & 6a-6 \end{pmatrix}$$

$AB = (6a-6)I \Rightarrow A^{-1} = \dfrac{1}{6a-6}B$

so $A^{-1} = \dfrac{1}{6a-6}\begin{pmatrix} 3a-1 & a+1 & -4 \\ 1 & 2a-1 & -2 \\ -3 & -3 & 6 \end{pmatrix}$, provided that $a \neq 1$.

(ii) $$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Therefore, provided $a \neq 1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{6a-6}\begin{pmatrix} 3a-1 & a+1 & -4 \\ 1 & 2a-1 & -2 \\ -3 & -3 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{6a-6}\begin{pmatrix} 4a-8 \\ 2a-4 \\ 6 \end{pmatrix}$$

So the solution is $x = \dfrac{2a-4}{3a-3}, y = \dfrac{a-2}{3a-3}, z = \dfrac{1}{a-1}$ provided $a \neq 1$.

If a = 1 then by adding the first two equations together and comparing to the third equation, it can be seen that there are no solutions.

5. $\begin{pmatrix} 2 & 1 & 3 \\ 1 & -2 & k+1 \\ k & 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 8 \end{pmatrix}$

(i) For k = -2, the determinant is zero so there is not a unique solution.

$2x + y + 3z = 5 \qquad (1)$

$x - 2y - z = 2 \qquad (2)$

$-2x + 4y + 2z = 8 \qquad (3)$

Equation (3) can be written as $x - 2y - z = -4$.

This is not consistent with equation (2), so there are no solutions.

This situation represents two parallel planes (equations 2 and 3), and a third plane (equation 1) which cuts across both.

(ii) For k = 2, there is a unique solution.

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -2 & 3 \\ 2 & 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 8 \end{pmatrix}$$

Inverse matrix $= -\frac{1}{4}\begin{pmatrix} -16 & 10 & 9 \\ 4 & -2 & -3 \\ 8 & -6 & -5 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -16 & 10 & 9 \\ 4 & -2 & -3 \\ 8 & -6 & -5 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \\ 8 \end{pmatrix}$$

$$= -\frac{1}{4}\begin{pmatrix} 12 \\ -8 \\ -12 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}$$

The unique solution is x = -3, y = 2, z = 3.

This situation represents three planes which meet at a point.

(iii) For k = 3, the determinant is zero so there is not a unique solution.

$2x + y + 3z = 5 \qquad (1)$

$x - 2y + 4z = 2 \qquad (2)$

$3x + 4y + 2z = 8 \qquad (3)$

(1) - 2×(2): $\qquad 5y - 5z = 1$

3×(2) – (3) $\qquad -10y + 10z = -2$

These equations are consistent.

This situation represents three planes which form a sheaf of planes, with a common line.

6. (i) $\det A = 2(k-2) - 1(0+3)$

$= 2k - 4 - 3$

$= 2k - 7$

The matrix is singular for $k = 3.5$.

(ii) $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -3 & 2 & 3 \end{pmatrix}$

From (i), for $k = 3$, det A = -1

Matrix of cofactors $= \begin{pmatrix} 1 & -3 & 3 \\ -2 & 3 & -4 \\ 1 & -2 & 2 \end{pmatrix}$

Inverse matrix $= -1\begin{pmatrix} 1 & -2 & 1 \\ -3 & 3 & -2 \\ 3 & -4 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ 3 & -3 & 2 \\ -3 & 4 & -2 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -3 & 2 & 3.5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ m \end{pmatrix}$

(1) $2x - z = 4$

(2) $y + z = 3$

(3) $-3x + 2y + 3.5z = m$

(1) $\Rightarrow z = 2x - 4$

(2) $\Rightarrow y = 3 - z = 3 - 2x + 4 = 7 - 2x$

Substituting into (3) gives $-3x + 14 - 4x + 7x - 14 = m$

$0 = m$

The solution is $x = t, y = 7 - 2t, z = 2t - 4$. This is the equation of a straight line which is the intersection line of three planes.

7. $\begin{vmatrix} 3 & -1 & 1 \\ 2 & 1 & -6 \\ 5 & -3 & k \end{vmatrix} = 3(k-18) - 1(-30-2k) + 1(-6-5)$

$= 5k - 35$

(A) For a unique solution, the determinant must be non-zero, so k can take any value other than 7, and r can take any value.
The geometric interpretation is three planes meeting at a point.

(B) For an infinite number of solutions, the determinant is zero so k = 7, and the equations must be consistent.

(1) $3x - y + z = 5$
(2) $2x + y - 6z = 5$
(3) $5x - 3y + 7z = r$

$(1)+(2) \Rightarrow 5x - 5z = 10 \quad \Rightarrow x - z = 2$

$3(2)+(3) \Rightarrow 11x - 11z = 15 + r$

For the equations to be consistent, $15 + r = 22 \quad \Rightarrow r = 7$

So $k = 7$ and $r = 7$.

The geometric interpretation is three planes forming a sheaf with a common line.

(c) For no solutions, the determinant is zero and the equations are inconsistent, so $k = 7$ and r can take any value other than 7.
The geometric interpretation is three planes forming a triangular prism.

8. (i) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 7 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 7 & 0 \end{vmatrix} = 1(0-21) + 1(0-2) = -23$$

$$\text{Matrix of cofactors} = \begin{pmatrix} -21 & 3 & -2 \\ 7 & -1 & -7 \\ -2 & -3 & 2 \end{pmatrix}$$

$$\text{Inverse matrix} = -\frac{1}{23}\begin{pmatrix} -21 & 7 & -2 \\ 3 & -1 & -3 \\ -2 & -7 & 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{23}\begin{pmatrix} -21 & 7 & -2 \\ 3 & -1 & -3 \\ -2 & -7 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -\frac{1}{23}\begin{pmatrix} -16 \\ -1 \\ -7 \end{pmatrix}$$

$x = \frac{16}{23}, y = \frac{1}{23}, z = \frac{7}{23}$

(ii) (1) $x + z = 1$
(2) $2y + 3z = 1$
(3) $x + 7y = 1$

(3)-(1) gives $7y - z = 0 \quad \Rightarrow z = 7y$

Substituting into (2) gives $2y + 21y = 1$
$y = \frac{1}{23}, z = \frac{7}{23}, x = \frac{16}{23}$