Section 2: The inverse of a 3×3 matrix

Solutions to Exercise level 2

1. (i) $$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & x \end{vmatrix} = 1(3x-16)+2(12-2x)+3(8-9)$$
$$= 3x - 16 + 24 - 4x - 3$$
$$= 5 - x$$
$$|\mathbf{B}| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 3 & 2 \\ 5 & 4 & x \end{vmatrix} = 3(3x-8)+2(10-4x)+1(16-15)$$
$$= 9x - 24 + 20 - 8x + 1$$
$$= x - 3$$

(ii) $|\mathbf{A}| + |\mathbf{B}| = 5 - x + x - 3 = 2$
so $|\mathbf{A}| + |\mathbf{B}| \neq 0$ for any values of x

(iii) $|\mathbf{A}| - |\mathbf{B}| = 5 - x - (x-3) = 8 - 2x$
so $|\mathbf{A}| + |\mathbf{B}| = 0$ when $x = 4$

2. $$\begin{pmatrix} 3 & -1 & 5 \\ 2 & 0 & -4 \\ 1 & -2 & 0 \end{pmatrix} \begin{matrix} 3 & -1 \\ 2 & 0 \\ 1 & -2 \end{matrix}$$
$$\begin{matrix} 3 & -1 & 5 & 3 & -1 \\ 2 & 0 & -4 & 2 & 0 \end{matrix}$$

Cofactors are:

$$\begin{vmatrix} 0 & -4 \\ -2 & 0 \end{vmatrix} = 0 - 8 \qquad \begin{vmatrix} -4 & 2 \\ 0 & 1 \end{vmatrix} = -4 - 0 \qquad \begin{vmatrix} 2 & 0 \\ 1 & -2 \end{vmatrix} = -4 - 0$$
$$= -8 \qquad\qquad = -4 \qquad\qquad = -4$$

$$\begin{vmatrix} -2 & 0 \\ -1 & 5 \end{vmatrix} = -10 - 0 \qquad \begin{vmatrix} 0 & 1 \\ 5 & 3 \end{vmatrix} = 0 - 5 \qquad \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = -1 - (-6)$$
$$= -10 \qquad\qquad = -5 \qquad\qquad = 5$$

$$\begin{vmatrix} -1 & 5 \\ 0 & -4 \end{vmatrix} = 4 - 0 \qquad \begin{vmatrix} 5 & 3 \\ -4 & 2 \end{vmatrix} = 10 - (-12) \qquad \begin{vmatrix} 3 & -1 \\ 2 & 0 \end{vmatrix} = 0 - (-2)$$
$$= 4 \qquad\qquad = 22 \qquad\qquad = 2$$

Matrix of cofactors is $\begin{pmatrix} -8 & -4 & -4 \\ -10 & -5 & 5 \\ 4 & 22 & 2 \end{pmatrix}$

Expanding by second row: Determinant $= 2\times -10 + 0 - 4\times 5$

$= -20 - 20$

$= -40$

$$\text{Inverse matrix} = -\frac{1}{40}\begin{pmatrix} -8 & -10 & 4 \\ -4 & -5 & 22 \\ -4 & 5 & 2 \end{pmatrix} = \frac{1}{40}\begin{pmatrix} 8 & 10 & -4 \\ 4 & 5 & -22 \\ 4 & -5 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -1 & 5 \\ 2 & 0 & -4 \\ 1 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & 5 \\ 2 & 0 & -4 \\ 1 & -2 & 0 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 10 \\ -1 \end{pmatrix} = \frac{1}{40}\begin{pmatrix} 8 & 10 & -4 \\ 4 & 5 & -22 \\ 4 & -5 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 10 \\ -1 \end{pmatrix} = \frac{1}{40}\begin{pmatrix} 120 \\ 80 \\ -30 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

The coordinates of P are (3, 2, -1).

3. $(\mathrm{ABC})^{-1}(\mathrm{ABC}) = \mathrm{I}$

Post-multiplying both sides:

$(\mathrm{ABC})^{-1}\mathrm{ABCC}^{-1} = \mathrm{C}^{-1}$

$(\mathrm{ABC})^{-1}\mathrm{AB} = \mathrm{C}^{-1}$

$(\mathrm{ABC})^{-1}\mathrm{ABB}^{-1} = \mathrm{C}^{-1}\mathrm{B}^{-1}$

$(\mathrm{ABC})^{-1}\mathrm{A} = \mathrm{C}^{-1}\mathrm{B}^{-1}$

$(\mathrm{ABC})^{-1}\mathrm{AA}^{-1} = \mathrm{C}^{-1}\mathrm{B}^{-1}\mathrm{A}^{-1}$

$(\mathrm{ABC})^{-1} = \mathrm{C}^{-1}\mathrm{B}^{-1}\mathrm{A}^{-1}$

4. If A is singular then det A = 0, and if B is singular then det B = 0.
Since det(AB) = det A × det B, then det(AB) is zero if and only if at least one of det A and det B is zero. If neither det A nor det B are zero, then det (AB) is always non-zero.

×	A singular	A non-singular
B singular	AB singular	AB singular
B non-singular	AB singular	AB non-singular

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5. (i) $P = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & -4 \\ 0 & 2 & -19 \end{pmatrix}$

$\det P = 1(-19+8)+3(4-0) = 1$

$\text{Matrix of cofactors} = \begin{pmatrix} -11 & 38 & 4 \\ 6 & -19 & -2 \\ -3 & 10 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} -11 & 6 & -3 \\ 38 & -19 & 10 \\ 4 & -2 & 1 \end{pmatrix}$

$Q = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$

$\det Q = \frac{1}{2}(0 - \frac{1}{12}) = -\frac{1}{24}$

$\text{Matrix of cofactors} = \begin{pmatrix} -\frac{1}{12} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} \\ 0 & -\frac{1}{6} & 0 \end{pmatrix}$

$Q^{-1} = -24\begin{pmatrix} -\frac{1}{12} & 0 & 0 \\ 0 & 0 & -\frac{1}{6} \\ 0 & -\frac{1}{8} & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{pmatrix}$

(ii) $PQ = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} & 0 \\ 1 & -1 & \frac{1}{3} \\ 0 & -\frac{19}{4} & \frac{2}{3} \end{pmatrix}$

$(PQ)^{-1} = \begin{pmatrix} -22 & 12 & -6 \\ 16 & -8 & 4 \\ 114 & -57 & 30 \end{pmatrix}$

$Q^{-1}P^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} -11 & 6 & -3 \\ 38 & -19 & 10 \\ 4 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -22 & 12 & -6 \\ 16 & -8 & 4 \\ 114 & -57 & 30 \end{pmatrix}$

