# Section 2: The inverse of a 3×3 matrix

## Solutions to Exercise level 1

1. (i) $\begin{pmatrix} 1 & 0 & 2 \\ -3 & 1 & 4 \\ -1 & -3 & -2 \end{pmatrix}\begin{matrix} 1 & 0 \\ -3 & 1 \\ -1 & -3 \end{matrix}$

Expanding by the first row: Determinant $= 1\begin{vmatrix} 1 & 4 \\ -3 & -2 \end{vmatrix} + 0 + 2\begin{vmatrix} -3 & 1 \\ -1 & -3 \end{vmatrix}$

$= 1(-2-(-12)) + 2(9-(-1))$

$= 10 + 20$

$= 30$

(ii) $\begin{pmatrix} 2 & 3 & -5 \\ 1 & -2 & -4 \\ 0 & 3 & 0 \end{pmatrix}\begin{matrix} 2 & 3 \\ 1 & -2 \\ 0 & 3 \end{matrix}$

$\begin{matrix} 2 & 3 & -5 & 2 & 3 \\ 1 & -2 & -4 & 1 & -2 \end{matrix}$

Expanding by the third row: Determinant $= 0 + 3\begin{vmatrix} -5 & 2 \\ -4 & 1 \end{vmatrix} + 0$

$= 3(-5-(-8))$

$= 9$

(iii) $\begin{pmatrix} -3 & 1 & 6 \\ -2 & 0 & k \\ 1 & -1 & 4 \end{pmatrix}\begin{matrix} -3 & 1 \\ -2 & 0 \\ 1 & -2 \end{matrix}$

$\begin{matrix} -3 & 1 & 6 & -3 & 1 \\ -2 & 0 & k & -2 & 0 \end{matrix}$

Expanding by the second column:

Determinant $= 1\begin{vmatrix} k & -2 \\ 4 & 1 \end{vmatrix} + 0 - 1\begin{vmatrix} 6 & -3 \\ k & -2 \end{vmatrix}$

$= 1(k-(-8)) - 1(-12-(-3k))$

$= k + 8 + 12 - 3k$

$= 20 - 2k$

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2. $P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}$

Expanding by the third row: $|P| = 3\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 3(0-2) = -6$

$Q = \begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & 0 \\ 4 & 0 & 0 \end{pmatrix}$

Expanding by the third row: $|Q| = 4\begin{vmatrix} 2 & 2 \\ 3 & 0 \end{vmatrix} = 4(0-6) = -24$

$$PQ = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & 0 \\ 4 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 5 & 2 \\ 10 & 10 & 4 \\ 6 & 6 & 6 \end{pmatrix}$$

$$QP = \begin{pmatrix} 2 & 2 & 2 \\ 3 & 3 & 0 \\ 4 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 6 & 2 \\ 9 & 9 & 3 \\ 4 & 4 & 4 \end{pmatrix}$$

$$|PQ| = 9\begin{vmatrix} 10 & 4 \\ 6 & 6 \end{vmatrix} + 5\begin{vmatrix} 4 & 10 \\ 6 & 6 \end{vmatrix} + 2\begin{vmatrix} 10 & 10 \\ 6 & 6 \end{vmatrix}$$
$$= 9(60-24) + 5(24-60) + 2(60-60)$$
$$= 144$$

$$|QP| = 12\begin{vmatrix} 9 & 3 \\ 4 & 4 \end{vmatrix} + 6\begin{vmatrix} 3 & 9 \\ 4 & 4 \end{vmatrix} + 2\begin{vmatrix} 9 & 9 \\ 4 & 4 \end{vmatrix}$$
$$= 12(36-12) + 6(12-36) + 2(36-36)$$
$$= 144$$

$\det(P) \times \det(Q) = -6 \times -24 = 144$

so $\det(PQ) = \det(QP) = \det(P) \times \det(Q)$

3. (i) $M = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 0 & 4 \\ -1 & 0 & -3 \end{pmatrix} \begin{matrix} 3 & 1 \\ 2 & 0 \\ -1 & 0 \end{matrix}$

Expanding by second column: $\det M = 1\begin{vmatrix} 4 & 2 \\ -3 & -1 \end{vmatrix} = -4-(-6) = 2$

Volume of image $= 5 \times 2 = 10\,\text{cm}^3$

(ii) Volume factor $= \dfrac{320}{5} = 64$

$(\det M)^n = 64 \quad \Rightarrow 2^n = 64 \quad \Rightarrow n = 6$

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4. (i) $\begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & -2 \\ 4 & 1 & 0 \end{pmatrix} \begin{matrix} 3 & 0 \\ 2 & 1 \\ 4 & 1 \end{matrix}$

$\begin{matrix} 3 & 0 & 1 & 3 & 0 \\ 2 & 1 & -2 & 2 & 1 \end{matrix}$

Expanding by first row: Determinant $= 3\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix}$

$= 3(0-(-2)) + 1(2-4)$

$= 6-2$

$= 4$

Cofactors are:

$\begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} = 0-(-2) = 2$ $\quad \begin{vmatrix} -2 & 2 \\ 0 & 4 \end{vmatrix} = -8-0 = -8$ $\quad \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} = 2-4 = -2$

$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1-0 = 1$ $\quad \begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} = 0-4 = -4$ $\quad \begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} = 0-3 = -3$

$\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 0-1 = -1$ $\quad \begin{vmatrix} 1 & 3 \\ -2 & 2 \end{vmatrix} = 2-(-6) = 8$ $\quad \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix} = 3-0 = 3$

Matrix of cofactors is $\begin{pmatrix} 2 & -8 & -2 \\ 1 & -4 & -3 \\ -1 & 8 & 3 \end{pmatrix}$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 1 & -1 \\ -8 & -4 & 8 \\ -2 & -3 & 3 \end{pmatrix}$$

(ii) $\begin{pmatrix} 1 & 2 & -3 \\ -2 & 1 & 1 \\ 3 & -1 & 0 \end{pmatrix} \begin{matrix} 1 & 2 \\ -2 & 1 \\ 3 & -1 \end{matrix}$

$\begin{matrix} 1 & 2 & -3 & 1 & 2 \\ -2 & 1 & 1 & -2 & 1 \end{matrix}$

Expanding by third row: Determinant $= 3\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} -3 & 1 \\ 1 & -2 \end{vmatrix}$

$= 3(2-(-3)) - 1(6-1)$

$= 15-5$

$= 10$

Cofactors are:

 

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$$\begin{vmatrix}1 & 1\\-1 & 0\end{vmatrix} = 0-(-1) = 1 \qquad \begin{vmatrix}1 & -2\\0 & 3\end{vmatrix} = 3-0 = 3 \qquad \begin{vmatrix}-2 & 1\\3 & -1\end{vmatrix} = 2-3 = -1$$

$$\begin{vmatrix}-1 & 0\\2 & -3\end{vmatrix} = 3-0 = 3 \qquad \begin{vmatrix}0 & 3\\-3 & 1\end{vmatrix} = 0-(-9) = 9 \qquad \begin{vmatrix}3 & -1\\1 & 2\end{vmatrix} = 6-(-1) = 7$$

$$\begin{vmatrix}2 & -3\\1 & 1\end{vmatrix} = 2-(-3) = 5 \qquad \begin{vmatrix}-3 & 1\\1 & -2\end{vmatrix} = 6-1 = 5 \qquad \begin{vmatrix}1 & 2\\-2 & 1\end{vmatrix} = 1-(-4) = 5$$

$$\text{Matrix of cofactors} = \begin{pmatrix}1 & 3 & -1\\3 & 9 & 7\\5 & 5 & 5\end{pmatrix}$$

$$B^{-1} = \frac{1}{10}\begin{pmatrix}1 & 3 & 5\\3 & 9 & 5\\-1 & 7 & 5\end{pmatrix}$$

(iii) $$(AB)^{-1} = B^{-1}A^{-1} = \frac{1}{10}\begin{pmatrix}1 & 3 & 5\\3 & 9 & 5\\-1 & 7 & 5\end{pmatrix} \times \frac{1}{4}\begin{pmatrix}2 & 1 & -1\\-8 & -4 & 8\\-2 & -3 & 3\end{pmatrix}$$

$$= \frac{1}{40}\begin{pmatrix}-32 & -26 & 38\\-76 & -48 & 84\\-68 & -44 & 72\end{pmatrix} = \begin{pmatrix}-0.8 & -0.65 & 0.95\\-1.9 & -1.2 & 2.1\\-1.7 & -1.1 & 1.8\end{pmatrix}$$

(iv) $$(BA)^{-1} = A^{-1}B^{-1} = \frac{1}{4}\begin{pmatrix}2 & 1 & -1\\-8 & -4 & 8\\-2 & -3 & 3\end{pmatrix} \times \frac{1}{10}\begin{pmatrix}1 & 3 & 5\\3 & 9 & 5\\-1 & 7 & 5\end{pmatrix}$$

$$= \frac{1}{40}\begin{pmatrix}6 & 8 & 10\\-28 & -4 & -20\\-14 & -12 & -10\end{pmatrix} = \begin{pmatrix}0.15 & 0.2 & 0.25\\-0.7 & -0.1 & -0.5\\-0.35 & -0.3 & -0.25\end{pmatrix}$$

5. (i) $\begin{pmatrix} 1 & -3 & 2 \\ 0 & 2 & -2 \\ -1 & 3 & k \end{pmatrix} \begin{matrix} 1 & -3 \\ 0 & 2 \\ -1 & 3 \end{matrix}$

$\begin{matrix} 1 & -3 & 2 & 1 & -3 \\ 0 & 2 & -2 & 0 & 2 \end{matrix}$

Expanding by the first column: determinant $= 1\begin{vmatrix} 2 & -2 \\ 3 & k \end{vmatrix} - 1\begin{vmatrix} -3 & 2 \\ 2 & -2 \end{vmatrix}$

$= 2k - (-6) - (6-4)$

$= 2k + 6 - 2$

$= 2k + 4$

If the matrix is singular, $2k + 4 = 0$

$k = -2$

(ii) Cofactors are:

$\begin{vmatrix} 2 & -2 \\ 3 & k \end{vmatrix} = 2k - (-6) = 2k + 6$ $\quad \begin{vmatrix} -2 & 0 \\ k & -1 \end{vmatrix} = 2 - 0 = 2$ $\quad \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} = 0 - (-2) = 2$

$\begin{vmatrix} 3 & k \\ -3 & 2 \end{vmatrix} = 6 - (-3k) = 6 + 3k$ $\quad \begin{vmatrix} k & -1 \\ 2 & 1 \end{vmatrix} = k - (-2) = k + 2$ $\quad \begin{vmatrix} -1 & 3 \\ 1 & -3 \end{vmatrix} = 3 - 3 = 0$

$\begin{vmatrix} -3 & 2 \\ 2 & -2 \end{vmatrix} = 6 - 4 = 2$ $\quad \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} = 0 - (-2) = 2$ $\quad \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

Matrix of cofactors $= \begin{pmatrix} 2k+6 & 2 & 2 \\ 3k+6 & k+2 & 0 \\ 2 & 2 & 2 \end{pmatrix}$

$$M^{-1} = \frac{1}{2k+4}\begin{pmatrix} 2k+6 & 3k+6 & 2 \\ 2 & k+2 & 2 \\ 2 & 0 & 2 \end{pmatrix}$$

6. A MINOR together with its SIGN gives the COFACTOR. The TRANSPOSE of the COFACTOR matrix gives the ADJUGATE matrix. The ADJUGATE matrix divided by the DETERMINANT gives the INVERSE matrix.