Section 1: Determinants and inverses

Solutions to Exercise level 3

1. (i) $\det A = ad - bc$

$\det B = eh - fg$

$\det A \det B = (ad - bc)(eh - fg)$

$= adeh - bceh - adfg + bcfg$

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$\det AB = (ae + bg)(cf + dh) - (af + bh)(ce + dg)$

$= \cancel{aecf} + bgcf + aedh + \cancel{bgdh} - \cancel{afce} - bhce - afdg - \cancel{bhdg}$

$= bgcf + aedh - bhce - afdg$

so $\det(AB) = \det A \det B$.

(ii) (a) $M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\det\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda^2$

$\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1$

$\det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 1 + 1 = 2$

so $\lambda^2 \times 1 = 2$

Since $\lambda > 0$, $\lambda = \sqrt{2}$

(c) $\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} \alpha\sqrt{2} & \beta\sqrt{2} \\ \gamma\sqrt{2} & \delta\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

so $\alpha = \gamma = \delta = \frac{1}{\sqrt{2}}$ and $\beta = -\frac{1}{\sqrt{2}}$

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$$\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

$$\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}\begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & \cos 45^\circ\end{pmatrix}=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

$\begin{pmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{pmatrix}$ is an enlargement centre O scale factor $\sqrt{2}$

$\begin{pmatrix}\cos 45^\circ & -\sin 45^\circ\\ \sin 45^\circ & \cos 45^\circ\end{pmatrix}$ is an anticlockwise rotation of 45° about O

So T is an anticlockwise rotation of 45° about O followed by an enlargement centre O scale factor $\sqrt{2}$.

2. (i) $\det \mathbf{B} = -2\lambda - 0 = -2\lambda$

Since $\lambda \neq 0$, $\det \mathbf{B} \neq 0$ and so $\mathbf{B}^{-1}$ exists.

(ii) $\mathbf{B}^{-1} = -\frac{1}{2\lambda}\begin{pmatrix}\lambda & 0\\ -1 & -2\end{pmatrix} = \frac{1}{2\lambda}\begin{pmatrix}-\lambda & 0\\ 1 & 2\end{pmatrix}$

$$\mathbf{B}^{-1}\mathbf{AB} = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$$

$$\frac{1}{2\lambda}\begin{pmatrix}-\lambda & 0\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 3\end{pmatrix}\begin{pmatrix}-2 & 0\\ 1 & \lambda\end{pmatrix}=\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$$

$$\begin{pmatrix}-\lambda & 0\\ 1 & 2\end{pmatrix}\begin{pmatrix}-2 & 0\\ 1 & 3\lambda\end{pmatrix}=2\lambda\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$$

$$\begin{pmatrix}2\lambda & 0\\ 0 & 6\lambda\end{pmatrix}=\begin{pmatrix}2\lambda a & 0\\ 0 & 2\lambda b\end{pmatrix}$$

So $a = 1, b = 3$ and λ can take any value except 0.

(iii)
$$\begin{aligned}(\mathbf{B}^{-1}\mathbf{AB})^2 &= (\mathbf{B}^{-1}\mathbf{AB})(\mathbf{B}^{-1}\mathbf{AB})\\ &= \mathbf{B}^{-1}\mathbf{ABB}^{-1}\mathbf{AB}\\ &= \mathbf{B}^{-1}\mathbf{AIAB}\\ &= \mathbf{B}^{-1}\mathbf{AAB}\\ &= \mathbf{B}^{-1}\mathbf{A}^2\mathbf{B}\end{aligned}$$

(iv) $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}^2 = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}=\begin{pmatrix}a^2 & 0\\ 0 & b^2\end{pmatrix}$

It's easy to see that $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}^{100} = \begin{pmatrix}a^{100} & 0\\ 0 & b^{100}\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 3^{100}\end{pmatrix}$

$$\text{Also } (B^{-1}AB)^3 = (B^{-1}AB)(B^{-1}AB)^2$$
$$= B^{-1}ABB^{-1}A^2B$$
$$= B^{-1}AA^2B$$
$$= B^{-1}A^3B$$

and so on, hence $(B^{-1}AB)^{100} = B^{-1}A^{100}B$

$$\text{But } B^{-1}AB = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\text{so } \left(B^{-1}AB\right)^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}^{100} = \begin{pmatrix} 1 & 0 \\ 0 & 3^{100} \end{pmatrix}$$

$$B^{-1}A^{100}B = \begin{pmatrix} 1 & 0 \\ 0 & 3^{100} \end{pmatrix}$$

$$BB^{-1}A^{100}BB^{-1} = B\begin{pmatrix} 1 & 0 \\ 0 & 3^{100} \end{pmatrix}B^{-1}$$

$$A^{100} = \begin{pmatrix} -2 & 0 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^{100} \end{pmatrix}\frac{1}{2\lambda}\begin{pmatrix} -\lambda & 0 \\ 1 & 2 \end{pmatrix}$$
$$= \frac{1}{2\lambda}\begin{pmatrix} -2 & 0 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} -\lambda & 0 \\ 3^{100} & 2\times 3^{100} \end{pmatrix}$$
$$= \frac{1}{2\lambda}\begin{pmatrix} 2\lambda & 0 \\ \lambda - \lambda\times 3^{100} & 2\lambda\times 3^{100} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{1}{2} - \frac{1}{2}\times 3^{100} & 3^{100} \end{pmatrix}$$

3. (i) $\det\begin{pmatrix} -2-\alpha & \sqrt{3} \\ \sqrt{3} & -\alpha \end{pmatrix} = 0$

$$-\alpha(-2-\alpha) - 3 = 0$$
$$\alpha^2 + 2\alpha - 3 = 0$$
$$(\alpha+3)(\alpha-1) = 0$$
$$\alpha = -3 \text{ or } \alpha = 1$$

(ii) $\alpha_1 = -3$

$$\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{pmatrix}\begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \cos\theta_1 + \sqrt{3}\sin\theta_1 \\ \sqrt{3}\cos\theta_1 + 3\sin\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \cos\theta_1 + \sqrt{3}\sin\theta_1 = 0$$

$$\Rightarrow \sqrt{3}\sin\theta_1 = -\cos\theta_1$$

$$\Rightarrow \tan\theta_1 = -\frac{1}{\sqrt{3}} \Rightarrow \theta_1 = \frac{5\pi}{6}$$

$$\alpha_2 = 1$$

$$\begin{pmatrix} -3 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3\cos\theta_2 + \sqrt{3}\sin\theta_2 \\ \sqrt{3}\cos\theta_2 - \sin\theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \sqrt{3}\cos\theta_2 - \sin\theta_2 = 0$$

$$\Rightarrow \sin\theta_2 = \sqrt{3}\cos\theta_2$$

$$\Rightarrow \tan\theta_2 = \sqrt{3} \Rightarrow \theta_2 = \frac{\pi}{3}$$

(iii) $\mathbf{P} = \begin{pmatrix} \cos\frac{5\pi}{6} & \cos\frac{\pi}{3} \\ \sin\frac{5\pi}{6} & \sin\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$$\mathbf{P}^2 = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} + \frac{1}{4} & -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} & \frac{1}{4} + \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So when n is even, $\mathbf{P}^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and when n is odd, $\mathbf{P}^n = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$