Section 1: Introduction to matrices

Solutions to Exercise level 3

1. $\begin{pmatrix} a & 4 \\ 5 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & b \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ -1 & 1 \end{pmatrix}$.

$\begin{pmatrix} a+c & 4+bc \\ 5-2c & 1 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ -1 & 1 \end{pmatrix}$

Equating elements: $5-2c=-1 \quad \Rightarrow c=3$

$4+bc=1 \quad \Rightarrow 4+3b=1 \quad \Rightarrow b=-1$

$a+c=5 \quad \Rightarrow a+3=5 \quad \Rightarrow a=2$

So $a=2, b=-1, c=3$

2. (i) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix}$

so the product of two diagonal matrices is also diagonal.

(ii) $A^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$

Generalising, $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$.

3. (i) $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

$A^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(ii) $A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

(iii) $A^k A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$.

Multiplying the expression for A^k by A gives the same form of matrix for A^{k+1}. This means that if the matrix used A^k is correct, the one for A^{k+1} is also correct. Since we know that A^1, A^2 and A^3 all have this form, then it must be true for all positive integer values of k.

(This is an example of proof by induction, which you will meet more formally in a later topic).

(iv) $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} + a\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$

$\begin{pmatrix} 1+a+b & n+a \\ 0 & 1+a+b \end{pmatrix} = 0$

$n + a = 0 \quad \Rightarrow a = -n$

$1 + a + b = 0 \quad \Rightarrow 1 - n + b = 0 \quad \Rightarrow b = n - 1$