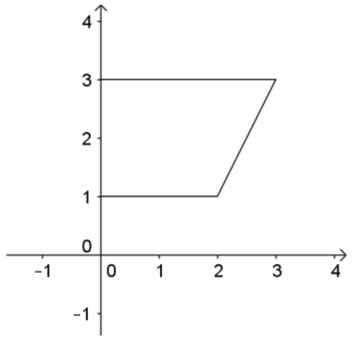
Section 1: Volumes of revolution

Exercise level 3 solutions

1.



Gradient of the line from (2, 1) to (3, 3) $= \frac{3-1}{3-2} = 2$

Equation of line is $y - 1 = 2(x-2)$

$y = 2x - 3$

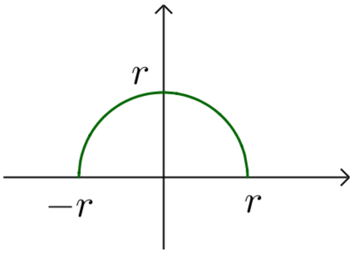
So need to rotate this line from $y = 1$ to $y = 3$.

$y = 2x - 3 \quad \Rightarrow x = \frac{1}{2}(y+3)$

$$\begin{aligned}\text{Volume} &= \pi\int_1^3 x^2 dy \\ &= \tfrac{1}{4}\pi\int_1^3 (y+3)^2 dy \\ &= \tfrac{1}{4}\pi\int_1^3 (y^2+6y+9) dy \\ &= \tfrac{1}{4}\pi\left[\tfrac{1}{3}y^3+3y^2+9y\right]_1^3 \\ &= \tfrac{1}{4}\pi\left(9+27+27-(\tfrac{1}{3}+3+9)\right) \\ &= \tfrac{38}{3}\pi\end{aligned}$$

2. A circle has equation $x^2 + y^2 = r^2$

The semicircle shown below is $y = \sqrt{r^2 - x^2}$



If this is rotated through 360° about the x-axis, this results in a sphere of radius r.

$$\begin{aligned}\text{Volume} &= \pi\int_{-r}^{r} y^2\,dx \\ &= \pi\int_{-r}^{r}(r^2 - x^2)\,dx \\ &= \pi\left[(r^2x - \tfrac{1}{3}x^3\right]_{-r}^{r} \\ &= \pi\left(r^3 - \tfrac{1}{3}r^3 - (-r^3 - \tfrac{1}{3}r^3)\right) \\ &= \tfrac{4}{3}\pi r^3\end{aligned}$$

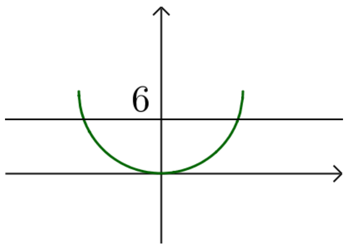
3. Use a circle with radius 9 and centre (0, 9).

The equation of this circle is $x^2 + (y-9)^2 = 81$

$$(y-9)^2 = 81 - x^2$$

The bottom half of the circle is given by $y - 9 = -\sqrt{81 - x^2}$

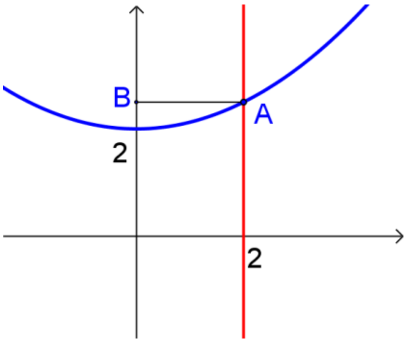
$$y = 9 - \sqrt{81 - x^2}$$



So the region that needs to be rotated is the part of this curve between $y = 0$ and $y = 6$.

$$\begin{aligned}\text{Volume} &= \pi\int_{0}^{6} x^2\,dy \\ &= \pi\int_{0}^{6}(81 - (y-9)^2)\,dy \\ &= \pi\int_{0}^{6}(81 - (y^2 - 18y + 81))\,dy \\ &= \pi\int_{0}^{6}(-y^2 + 18y)\,dy \\ &= \pi\left[-\tfrac{1}{3}y^3 + 9y^2\right]_{0}^{6} \\ &= \pi(-72 + 324) \\ &= 252\pi\end{aligned}$$

4.



At point A, $x = 2$ and $y = \frac{1}{8} \times 2^2 + 2 = 2.5$

So B is $(0, 2.5)$

The required volume can be found by finding the volume of the cylinder formed by rotating the line $x = 2$ from $y = 0$ to $y = 2.5$ about the y-axis, and then subtracting the volume formed by rotating the curve $y = \frac{1}{8}x^2 + 2$ from $y = 2$ to $y = 2.5$.

The cylinder has radius 2 and height 2.5.

Volume of cylinder $= \pi r^2 h = \pi \times 2^2 \times 2.5 = 10\pi$

$y = \frac{1}{8}x^2 + 2 \quad \Rightarrow \frac{1}{8}x^2 = y - 2 \quad \Rightarrow x^2 = 8y - 16$

Volume of solid formed by rotating curve is given by

$$\begin{aligned}
\text{Volume} &= \pi \int_2^{2.5} x^2 \, dy \\
&= \pi \int_2^{2.5} (8y - 16) \, dy \\
&= \pi \left[4y^2 - 16y \right]_2^{2.5} \\
&= \pi (25 - 40 - (16 - 32)) \\
&= \pi
\end{aligned}$$

So the required volume $= 10\pi - \pi = 9\pi$