Section 1: Volumes of revolution

Solutions to Exercise level 2

1. $y = \sqrt{x-1} \quad \Rightarrow y^2 = x-1 \quad \Rightarrow x = y^2 + 1$

$$\text{Volume} = \int_1^2 \pi x^2 \, dy = \pi \int_1^2 (y^2+1)^2 \, dy$$
$$= \pi \int_1^2 (y^4 + 2y^2 + 1) \, dy$$
$$= \pi \left[\tfrac{1}{5}y^5 + \tfrac{2}{3}y^3 + y \right]_1^2$$
$$= \pi\left(\tfrac{32}{5} + \tfrac{16}{3} + 2\right) - \pi\left(\tfrac{1}{5} + \tfrac{2}{3} + 1\right)$$
$$= \pi\left(\tfrac{31}{5} + \tfrac{14}{3} + 1\right)$$
$$= \tfrac{178}{15}\pi$$

2. $y = kx$

$$\text{Volume} = \int_0^h \pi y^2 \, dx$$
$$= \pi \int_0^h (kx)^2 \, dx$$
$$= \pi k^2 \int_0^h x^2 \, dx$$
$$= \pi k^2 \left[\tfrac{1}{3}x^3 \right]_0^h$$
$$= \pi k^2 \left(\tfrac{1}{3}h^3 - 0 \right)$$
$$= \tfrac{1}{3}\pi k^2 h^3$$

When $x = h$, $y = r$.

$y = kx \quad r = kh \quad \Rightarrow k = \dfrac{r}{h}$

$$\text{Volume of cone} = \tfrac{1}{3}\pi\left(\frac{r}{h}\right)^2 h^3 = \tfrac{1}{3}\pi \times \frac{r^2}{h^2} \times h^3 = \tfrac{1}{3}\pi r^2 h$$

3. $y = \dfrac{1}{x}$

For a rotation about the x-axis:

$$\text{Volume} = \int_1^2 \pi y^2 \, dx = \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx$$
$$= \pi \int_1^2 x^{-2} \, dx$$
$$= \pi \left[-x^{-1} \right]_1^2$$
$$= \pi\left(-\tfrac{1}{2} + 1\right)$$
$$= \tfrac{1}{2}\pi$$

When $x = 1$, $y = 1$
When $x = 2$, $y = \frac{1}{2}$

$$x = \frac{1}{y}$$

For a rotation about the y-axis:

$$\begin{aligned}\text{Volume} &= \int_{\frac{1}{2}}^{1} \pi x^2 \, dy = \pi \int_{\frac{1}{2}}^{1} \left(\frac{1}{y}\right)^2 dy \\ &= \pi \int_{\frac{1}{2}}^{1} y^{-2} \, dy \\ &= \pi \left[-y^{-1}\right]_{\frac{1}{2}}^{1} \\ &= \pi(-1 + 2) \\ &= \pi\end{aligned}$$

So the volume of revolution about the y-axis is twice the volume of revolution about the x-axis.

4. P and Q are the points at which $y = 0$, so $\frac{x^2}{4} = 1 \quad \Rightarrow x^2 = 4 \quad \Rightarrow x = \pm 2$

The coordinates of P are (2, 0) and the coordinates of Q are (-2, 0).

$$\frac{x^2}{4} + y^2 = 1 \quad \Rightarrow y^2 = 1 - \frac{x^2}{4}$$

$$\begin{aligned}\text{Volume} &= \int_{-2}^{2} \pi y^2 \, dx \\ &= \pi \int_{-2}^{2} \left(1 - \frac{x^2}{4}\right) dx \\ &= \pi \left[x - \tfrac{1}{12}x^3\right]_{-2}^{2} \\ &= \pi\left(2 - \tfrac{8}{12}\right) - \pi\left(-2 + \tfrac{8}{12}\right) \\ &= \pi\left(4 - \tfrac{4}{3}\right) \\ &= \tfrac{8}{3}\pi\end{aligned}$$

5. At the points at which $x = 0$, $y^2 = 1 \quad \Rightarrow y = \pm 1$.

$$\frac{x^2}{4} + y^2 = 1 \quad \Rightarrow x^2 = 4\left(1 - y^2\right)$$

$$\begin{aligned}\text{Volume} &= \int_{-1}^{1} \pi x^2 \, dy \\ &= \pi \int_{-1}^{1} 4\left(1 - y^2\right) dy \\ &= 4\pi \left[y - \tfrac{1}{3} y^3 \right]_{-1}^{1} \\ &= 4\pi \left(1 - \tfrac{1}{3}\right) - 4\pi \left(-1 + \tfrac{1}{3}\right) \\ &= 4\pi \left(2 - \tfrac{2}{3}\right) \\ &= \tfrac{16}{3}\pi \end{aligned}$$

6. At intersections of line and curve,

$$\begin{aligned} x^3 &= x \\ x^3 - x &= 0 \\ x(x^2 - 1) &= 0 \\ x(x-1)(x+1) &= 0 \\ x &= 0 \text{ or } 1 \text{ or } -1 \end{aligned}$$

Since $x \geq 0$, the intersection points are (0, 0) and (1, 1).

(i) For a volume of revolution about the x-axis:

$$\begin{aligned}\text{Volume} &= \int_{0}^{1} \pi y^2 \, dx \\ &= \pi \int_{0}^{1} \left(x^2 - (x^3)^2\right) dx \\ &= \pi \int_{0}^{1} (x^2 - x^6) \, dx \\ &= \pi \left[\tfrac{1}{3} x^3 - \tfrac{1}{7} x^7 \right]_{0}^{1} \\ &= \pi \left(\tfrac{1}{3} - \tfrac{1}{7} - 0 \right) \\ &= \tfrac{4}{21}\pi \end{aligned}$$

(ii) For a volume of revolution about the y-axis:

$$y = x^3 \quad \Rightarrow x = y^{\frac{1}{3}}$$

$$\begin{aligned}\text{Volume} &= \int_{0}^{1} \pi x^2 \, dy \\ &= \pi \int_{0}^{1} \left((y^{\frac{1}{3}})^2 - y^2\right) dy \\ &= \pi \int_{0}^{1} (y^{\frac{2}{3}} - y^2) \, dy \\ &= \pi \left[\tfrac{3}{5} y^{\frac{5}{3}} - \tfrac{1}{3} y^3 \right]_{0}^{1} \\ &= \pi \left(\tfrac{3}{5} - \tfrac{1}{3} - 0 \right) \\ &= \tfrac{4}{15}\pi \end{aligned}$$