Section 2: Complex roots of polynomials

Solutions to Exercise level 3

1. The roots of the equation must be conjugate pairs.
   Let the roots be $p \pm iq$
   so the equation is $x = p \pm iq$

$$x - p = \pm iq$$
$$(x-p)^2 = (\pm iq)^2$$
$$x^2 - 2px + p^2 = -q^2$$
$$x^2 - 2px + p^2 + q^2 = 0$$

   So $a = -2p$ and $b = p^2 + q^2$

   For $\frac{1+i}{4}$, $p = \frac{1}{4}$ and $q = \frac{1}{4}$, so $2p$ is not an integer

   For $\frac{1+i\sqrt{11}}{2}$, $p = \frac{1}{2}$ and $q = \frac{1}{2}\sqrt{11}$, so $2p$ is an integer,
   and $p^2 + q^2 = \frac{1}{4} + \frac{11}{4} = \frac{12}{4} = 3$ which is an integer

   For $\frac{1}{2} + i$, $p = \frac{1}{2}$ and $q = 1$, so $p^2 + q^2 = \frac{1}{4} + 1$ which is not an integer

   For $\frac{1-i\sqrt{7}}{2}$, $p = \frac{1}{2}$ and $q = -\frac{\sqrt{7}}{2}$, so $2p$ is an integer,
   and $p^2 + q^2 = \frac{1}{4} + \frac{7}{4} = \frac{8}{4} = 2$ which is an integer

   So $\frac{1+i\sqrt{11}}{2}$ and $\frac{1-i\sqrt{7}}{2}$ can be a root of the equation.

2. (i) By inspection, $z = 1$ is a real root of the equation.

   (ii) $z^3 - az^2 + az - 1 = 0$
   $(z-1)(z^2 + (1-a)z + 1) = 0$
   For the other two roots to be complex, the discriminant of the quadratic must be negative: $(1-a)^2 - 4 \times 1 \times 1 < 0$

$$1 - 2a + a^2 - 4 < 0$$
$$a^2 - 2a - 3 < 0$$
$$(a-3)(a+1) < 0$$
$$-1 < a < 3$$

(iii) The roots of the quadratic factor are $z = \dfrac{-(1-a) \pm \sqrt{(a-3)(a+1)}}{2}$

$$= \dfrac{a-1 \pm i\sqrt{(3-a)(a+1)}}{2}$$

So for the complex roots, $x = \dfrac{a-1}{2}$ and $y = \dfrac{\sqrt{(3-a)(a+1)}}{2}$

$$x^2 + y^2 = \frac{(a-1)^2}{4} + \frac{(3-a)(a+1)}{4}$$
$$= \frac{a^2 - 2a + 1 + 3a - a^2 + 3 - a}{4}$$
$$= \frac{4}{4} = 1$$

so all the complex roots lie on the unit circle.

3. (i) Let $z = a + ib$ and $w = c + id$.

$$(z+w)^* = (a+ib+c+id)^*$$
$$= (a+c+i(b+d))^*$$
$$= a+c-i(b+d)$$
$$= a+c-ib-id$$
$$= a-ib+c-id$$
$$= z^* + w^*$$

$$(zw)^* = ((a+ib)(c+id))^*$$
$$= (ac+ibc+ibd-ad)^*$$
$$= (ac-bd+i(bc+ad))^*$$
$$= ac-bd-i(bc+ad)$$
$$z^*w^* = (a-ib)(c-id)$$
$$= ac-ibc-iad-bd$$
$$= ac-bd-i(bc+ad)$$

(ii) $q(z) = a_m z^m$

$$(q(z))^* = (a_m z^m)^*$$
$$= a_m (z^m)^*$$
$$= a_m (zz^{m-1})^*$$
$$= a_m z^* (z^{m-1})^*$$
$$= a_m z^* (zz^{m-2})^*$$
$$= a_m z^* z^* (z^{m-2})^*$$

and so on, ending up with

$$(q(z))^* = a_m z^* z^* \ldots z^* = a_m (z^*)^m = q(z^*)$$

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(iii) $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

$\left(p(z)\right)^* = \left(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\right)^*$

Since $(z + w)^* = z^* + w^*$,

$\left(p(z)\right)^* = \left(a_n z^n\right)^* + \left(a_{n-1} z^{n-1}\right)^* + \cdots + \left(a_1 z\right)^* + \left(a_0\right)^*$

Using the result from (ii),

$\left(p(z)\right)^* = a_n (z^*)^n + a_{n-1} (z^*)^{n-1} + \cdots + a_1 z^* + a_0 = p(z^*)$

This result is very powerful. It tells us that, if a complex number $z = z_0$ is such that $p(z_0) = 0$, then $\left(p(z_0)\right)^* = 0^* = 0$

But since $\left(p(z)\right)^* = p(z^*)$, this means that $p(z^*) = 0$

and therefore if $z = z_0$ is the root of a polynomial with real coefficients, then $z = z_0{}^*$ is also a root of the polynomial.

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