Section 2: Complex roots of polynomials

Solutions to Exercise level 2

1. $3 + i$ is a root, so $3 - i$ is also a root.

   Therefore a quadratic factor is $(z-3-i)(z-3+i) = (z-3)^2 + 1$

   $= z^2 - 6z + 10$

   $1 + 3i$ is a root, so $1 - 3i$ is also a root.

   Therefore a quadratic factor is $(z-1-3i)(z-1+3i) = (z-1)^2 + 9$

   $= z^2 - 2z + 10$

   So the equation is $(z^2 - 6z + 10)(z^2 - 2z + 10) = 0$

   $z^4 - 8z^3 + 32z^2 - 80z + 100 = 0$

2. Since $1+i$ is a root, $1-i$ is also a root.

   The sum of the roots is 0 since the coefficient of z^2 is zero

   so $1+i+1-i+\alpha = 0$

   $\alpha = -2$

   So the three roots are $1+i$, $1-i$ and -2.

   The product of the roots is $-2(1+i)(1-i) = -k$

   $\Rightarrow k = 2(1+1)$

   $\Rightarrow k = 4$

3. $z = -1 + i$

   $z^2 = (-1+i)^2 = 1 - 2i - 1 = -2i$

   $z^3 = -2i(-1+i) = 2i + 2 = 2 + 2i$

   $z^4 = (2+2i)(-1+i) = -2 - 2 = -4$

   Substituting into $z^4 - 2z^3 - z^2 + 2z + 10$:

   $-4 - 2(2+2i) - (-2i) + 2(-1+i) + 10$

   $= -4 - 4 - 4i + 2i - 2 + 2i + 10$

   $= 0$

   so $-1 + i$ is a root.

   Since $-1 + i$ is a root, $-1 - i$ is also a root

   So a quadratic factor is $(z+1-i)(z+1+i) = (z+1)^2 + 1$

   $= z^2 + 2z + 2$

   $z^4 - 2z^3 - z^2 + 2z + 10 = (z^2 + 2z + 2)(z^2 - 4z + 5)$

   The other factors are the roots of the quadratic equation $z^2 - 4z + 5 = 0$

$$z = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 5}}{2}$$
$$= \frac{4 \pm \sqrt{-4}}{2}$$
$$= \frac{4 \pm 2i}{2}$$
$$= 2 \pm i$$

So the other roots are -1 – i, 2 + i and 2 – i.

4. $1+2i$ is a root so $1-2i$ is another root

A quadratic factor is $(z-1-2i)(z-1+2i) = (z-1)^2 + 4$

$$= z^2 - 2z + 5$$

$$z^4 - 6z^3 + 18z^2 - 30z + 25 = 0$$

$$(z^2 - 2z + 5)(z^2 - 4z + 5) = 0$$

$$z^2 - 4z + 5 = 0 \Rightarrow z = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$$\Rightarrow z = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

So the roots are $z = 1 \pm 2i$ and $z = 2 \pm i$.

5. Since $p+qi$ is a root, $p-qi$ is also a root.

The sum of the roots is 0 since the coefficient of z² is zero

so $p + qi + p - qi + \alpha = 0$

$\alpha = -2p$

So the three roots are $p+qi$, $p-qi$ and $-2p$.

(i) $\alpha\beta\gamma = -b$

$-2p(p+qi)(p-qi) = -b$

$2p(p^2+q^2) = b$

(ii) $\sum \alpha\beta = a$

$-2p(p+qi) - 2p(p-qi) + (p-qi)(p+qi) = a$

$-2p^2 - 2p^2 + p^2 + q^2 = a$

$-3p^2 + q^2 = a$

(iii) From (ii), $q^2 = 3p^2 + a$

Substituting into the result from (i):

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$$2p(p^2+3p^2+a)=b$$
$$2p(4p^2+a)=b$$
$$8p^3+2ap-b=0$$

so p is a root of the equation $8x^3+2ax-b=0$