Section 1: Roots and coefficients

Solutions to Exercise level 3

1. Two of the roots of the cubic equation are -3 and 1.
Let the third root be α.

Sum of roots $= -3 + 1 + \alpha = -\frac{b}{a} = 0 \quad \Rightarrow \alpha = 2$

So $x^3 + ax + b = (x+3)(x-1)(x-2)$

$= (x^2 + 2x - 3)(x-2)$

$= x^3 + 2x^2 - 3x - 2x^2 - 4x + 6$

$= x^3 - 7x + 6$

So $a = -7$ and $b = 6$

$x^2 + ax + b = 0$

$x^2 - 7x + 6 = 0$

$(x-1)(x-6) = 0$

$x = 1$ or $x = 6$

2. (i) $x^3 + 5x + 3 = 0$

To find an equation with roots $1+a, 1+b$ and $1+c$, use a substitution $y = 1 + x$, so $x = y - 1$

$(y-1)^3 + 5(y-1) + 3 = 0$

$y^3 - 3y^2 + 3y - 1 + 5y - 5 + 3 = 0$

$y^3 - 3y^2 + 8y - 3 = 0$

Product of roots $= -\frac{d}{a} = 3$

so $(1+a)(1+b)(1+c) = 3$

(ii) To find an equation with roots $\frac{1}{1+a}, \frac{1}{1+b}$ and $\frac{1}{1+c}$, start from the equation found in (i) and use a substitution

$z = \frac{1}{y} \Rightarrow y = \frac{1}{z}$

$\left(\frac{1}{z}\right)^3 - 3\left(\frac{1}{z}\right)^2 + 8\left(\frac{1}{z}\right) - 3 = 0$

$1 - 3z + 8z^2 - 3z^3 = 0$

$3z^3 - 8z^2 + 3z - 1 = 0$

Sum of roots $= -\frac{b}{a} = \frac{8}{3}$

so $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=\frac{8}{3}$

3. $x^4-2x^3+ax^2+8x+b=0$

Sum of roots is 2, so $\alpha+\beta+\gamma+\delta=2$
$\alpha+\beta=0$ so $\gamma+\delta=2$
$\delta-\gamma=4\Rightarrow\delta=\gamma+4$ so $\gamma+\gamma+4=2\Rightarrow 2\gamma=-2\Rightarrow\gamma=-1$ and $\delta=3$

$\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta=-8$
We have $\gamma=-1$, $\delta=3$ and $\beta=-\alpha$
$\alpha^2+3\alpha-3\alpha-3\alpha^2=-8$
$-2\alpha^2=-8$
$\alpha^2=4$
$\alpha=\pm2$
so $\alpha=2$ and $\beta=-2$ (it doesn't matter which way round)

So equation is $(x-2)(x+2)(x+1)(x-3)=0$
$(x^2-4)(x^2-2x-3)=0$
$x^4-2x^3-3x^2-4x^2+8x+12=0$
$x^4-2x^3-7x^2+8x+12=0$
so $a=-7$ and $b=12$