Section 1: Roots and coefficients

Solutions to Exercise level 2

1. Let the roots of the equation $2x^2 - kx + k = 0$ be α and 2α.

   Sum of roots: $\alpha + 2\alpha = \frac{k}{2} \quad \Rightarrow \alpha = \frac{k}{6}$

   Product of roots: $\alpha \times 2\alpha = \frac{k}{2}$

   $$4\alpha^2 = k$$
   $$4\left(\frac{k}{6}\right)^2 = k$$
   $$k^2 = 9k$$
   $$k(k-9) = 0$$

   Since $k \neq 0$, the value of k must be 9.

2. Let the roots of the equation $x^2 + (7-p)x - p = 0$ be α and $\alpha + 5$.

   Sum of roots: $\alpha + \alpha + 5 = -(7-p)$

   $$2\alpha + 5 = -7 + p$$
   $$2\alpha = p - 12$$

   Product of roots: $\alpha(\alpha+5) = -p$

   $$\left(\frac{p-12}{2}\right)\left(\frac{p-12}{2} + 5\right) = -p$$
   $$(p-12)(p-12+10) = -4p$$
   $$(p-12)(p-2) = -4p$$
   $$p^2 - 14p + 24 = -4p$$
   $$p^2 - 10p + 24 = 0$$
   $$(p-4)(p-6) = 0$$

   $p = 4$ or $p = 6$.

3. (i) $\beta = 4\alpha$

   Sum of roots: $\alpha + 4\alpha = -\frac{b}{a} \quad \Rightarrow \alpha = -\frac{b}{5a}$

Product of roots: $\alpha \times 4\alpha = \frac{c}{a}$

$$4\left(-\frac{b}{5a}\right)^2 = \frac{c}{a}$$

$$\frac{4b^2}{25a^2} = \frac{c}{a}$$

$$4b^2 = 25ac$$

(ii) $\beta = \alpha + 1$

Sum of roots: $\alpha + \alpha + 1 = -\frac{b}{a} \quad \Rightarrow 2\alpha = -\frac{b}{a} - 1$

Product of roots: $\alpha(\alpha + 1) = \frac{c}{a}$

$$\left(-\frac{b}{2a} - \frac{1}{2}\right)\left(-\frac{b}{2a} - \frac{1}{2} + 1\right) = \frac{c}{a}$$

$$\left(\frac{b}{2a} + \frac{1}{2}\right)\left(\frac{b}{2a} - \frac{1}{2}\right) = \frac{c}{a}$$

$$\frac{b^2}{4a^2} - \frac{1}{4} = \frac{c}{a}$$

$$b^2 - a^2 = 4ac$$

$$a^2 = b^2 - 4ac$$

4. $\sum \alpha = -5$

$\alpha + 2\alpha + \alpha + 3 = -5$

$4\alpha = -8$

$\alpha = -2$

The roots of the equation are -2, -4 and 1.

$\sum \alpha\beta = h$

$(-2 \times -4) + (-4 \times 1) + (1 \times -2) = h$

$8 - 4 - 2 = h$

$h = 2$

$\alpha\beta\gamma = -k$

$-2 \times -4 \times 1 = -k$

$k = -8$

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5. Let the roots be $\frac{\alpha}{r}, \alpha, \alpha r$

$\alpha\beta\gamma = \frac{3}{24}$

$\frac{\alpha}{r} \times \alpha \times \alpha r = \frac{1}{8}$

$\alpha^3 = \frac{1}{8}$

$\alpha = \frac{1}{2}$

$\sum \alpha = -\frac{28}{24}$

$\frac{\alpha}{r} + \alpha + \alpha r = -\frac{7}{6}$

$\frac{1}{2}\left(\frac{1}{r} + 1 + r\right) = -\frac{7}{6}$

$3 + 3r + 3r^2 = -7r$

$3r^2 + 10r + 3 = 0$

$(3r+1)(r+3) = 0$

$r = -\frac{1}{3}$ or -3

Roots are $-\frac{1}{6}$, $\frac{1}{2}$ and $-\frac{3}{2}$

6. $\alpha\beta\gamma = -\frac{d}{a} \quad \Rightarrow \alpha \times \frac{1}{\alpha} \times \beta = \frac{9}{6} \quad \Rightarrow \beta = \frac{3}{2}$

$$\sum \alpha = -\frac{b}{a} \quad \Rightarrow \alpha + \frac{1}{\alpha} + \beta = -\frac{11}{6}$$

$$\Rightarrow \alpha + \frac{1}{\alpha} = -\frac{11}{6} - \frac{3}{2} = -\frac{10}{3}$$

$$\Rightarrow 3\alpha^2 + 10\alpha + 3 = 0$$

$$\Rightarrow (3\alpha+1)(\alpha+3) = 0$$

$$\Rightarrow \alpha = -\tfrac{1}{3} \text{ or } -3$$

The roots of the equation are $-\frac{1}{3}$, -3 and $\frac{3}{2}$.

$$\sum \alpha\beta = \frac{c}{a} \quad \Rightarrow \left(\alpha \times \frac{1}{\alpha}\right) + \left(\frac{1}{\alpha} \times \beta\right) + \alpha\beta = \frac{k}{6}$$

$$\Rightarrow \left(-\frac{1}{3} \times -3\right) + \left(-\frac{1}{3} \times \frac{3}{2}\right) + \left(-3 \times \frac{3}{2}\right) = \frac{k}{6}$$

$$\Rightarrow k = 6\left(1 - \frac{1}{2} - \frac{9}{2}\right) = -24$$

The value of k is -24.

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7. (i) Let $y = 2x$ so $x = \frac{y}{2}$.

Substituting into $x^3 - 2x^2 - x + 2 = 0$:

$$\left(\frac{y}{2}\right)^3 - 2\left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right) + 2 = 0$$

$$\frac{y^3}{8} - \frac{2y^2}{4} - \frac{y}{2} + 2 = 0$$

$$y^3 - 4y^2 - 4y + 16 = 0$$

(ii) Let $y = x - 3$ so $x = y + 3$

Substituting into $x^3 - 2x^2 - x + 2 = 0$:

$$(y+3)^3 - 2(y+3)^2 - (y+3) + 2 = 0$$

$$y^3 + 9y^2 + 27y + 27 - 2y^2 - 12y - 18 - y - 3 + 2 = 0$$

$$y^3 + 7y^2 + 14y + 8 = 0$$

8. $\sum \alpha = -a$

$$\alpha + \beta + \gamma + \delta = -a$$

$$2(\alpha + \beta) = -a$$

$$\alpha + \beta = -\frac{a}{2}$$

$\sum \alpha\beta = b$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = b$$

$$\alpha\beta + (\alpha + \beta)(\gamma + \delta) + \gamma\delta = b$$

$$\alpha\beta + \gamma\delta = b - (\alpha + \beta)^2 = b - \frac{a^2}{4}$$

$\sum \alpha\beta\gamma = -c$

$$\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -c$$

$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -c$$

$$\alpha\beta(\alpha + \beta) + \gamma\delta(\alpha + \beta) = -c$$

$$(\alpha\beta + \gamma\delta)(\alpha + \beta) = -c$$

$$\left(b - \frac{a^2}{4}\right)\left(-\frac{a}{2}\right) = -c$$

$$(4b - a^2)a = 8c$$

$$a^3 + 8c = 4ab$$