Section 1: Modulus and argument

Exercise level 3

1. (i) $|z-w|^2 = (z-w)(z-w)*$
$= (z-w)(z*-w*)$
$= zz*-zw*-wz*+ww*$

(ii) $|z|=2 \Rightarrow |z|^2 = 4 \Rightarrow zz* = 4$
Similarly $|w|=2 \Rightarrow |w|^2 = 4 \Rightarrow w\bar{w} = 4$
$|z+w|=3 \Rightarrow (z+w)(z+w)* = 9$
$\Rightarrow (z+w)(z*+w*) = 9$
$\Rightarrow zz*+zw*+z*w+ww* = 9$
$\Rightarrow 4+zw*+z*w+4 = 9$
$\Rightarrow zw*+z*w = 1$
From (i), $|z-w|^2 = zz*+ww*-(zw*+wz*)$
$= 4+4-1$
$= 7$
so $|z-w| = \sqrt{7}$

2. (i) $\frac{1}{z} = \frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2}$

(ii) As F is outside the circle with radius 1, then $|F|>1$.
So $|F|^2 = a^2+b^2 > 1$
$\left|\frac{1}{F}\right|^2 = \frac{a^2}{(a^2+b^2)^2} + \frac{b^2}{(a^2+b^2)^2} = \frac{a^2+b^2}{(a^2+b^2)^2} = \frac{1}{a^2+b^2}$
So $\left|\frac{1}{F}\right|^2 < 1$ and therefore $\frac{1}{F}$ lies inside the circle.
As F is in the first quadrant, $a>0$ and $b>0$.
So $\text{Re}\left(\frac{1}{F}\right)>0$ and $\text{Im}\left(\frac{1}{F}\right)<0$ and so $\frac{1}{F}$ is in the fourth quadrant.
So the reciprocal of F must be point C.

3. (i) Let $z = a+ib$ and $w = c+id$
$\frac{z}{w} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{ac+ibc-iad+bd}{c^2+d^2}$
$= \frac{ac+bd+i(bc-ad)}{c^2+d^2}$

$$\left(\frac{z}{w}\right)^* = \frac{ac+bd-i(bc-ad)}{c^2+d^2} = \frac{ac+bd+i(ad-bc)}{c^2+d^2}$$

$$\frac{z^*}{w^*} = \frac{a-ib}{c-id} = \frac{(a-ib)(c+id)}{(c-id)(c+id)} = \frac{ac-ibc+iad+bd}{c^2+d^2}$$

$$= \frac{ac+bd+i(ad-bc)}{c^2+d^2}$$

so $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$

(ii) $|z+w| = |z-w|$

$\Rightarrow (z+w)(z+w)^* = (z-w)(z-w)^*$

$\Rightarrow (z+w)(z^*+w^*) = (z-w)(z^*-w^*)$

$\Rightarrow zz^*+zw^*+z^*w+ww^* = zz^*-zw^*-z^*w+ww^*$

$\Rightarrow 2zw^*+2z^*w = 0$

$\Rightarrow zw^*+z^*w = 0$

As $w \neq 0$, we can divide through by ww^*

so $\frac{z}{w}+\frac{z^*}{w^*} = 0$

Using the result from (i), $\frac{z}{w}+\left(\frac{z}{w}\right)^* = 0$

But $\frac{z}{w}+\left(\frac{z}{w}\right)^* = \text{Re}\left(\frac{z}{w}\right)$, so $\frac{z}{w}$ lies on the imaginary axis and so the possible values of $\arg\left(\frac{z}{w}\right)$ are $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

4. (i) $z_1 = \frac{1}{2}(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$, $z_2 = 2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

$A = \frac{1}{2}\times 2\times\frac{1}{2}\sin\frac{\pi}{3}$

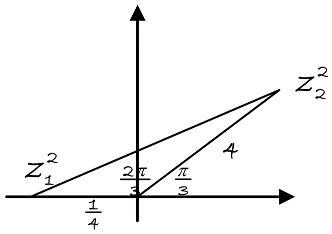
$= \frac{1}{4}\sqrt{3}$

(ii) $|z_1^2| = |z_1|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\arg(z_1^2) = \arg z_1 + \arg z_1 = \frac{\pi}{2}+\frac{\pi}{2} = \pi$

$|z_2^2| = |z_2|^2 = 2^2 = 4$

$\arg(z_2^2) = \arg z_2 + \arg z_2 = \frac{\pi}{6}+\frac{\pi}{6} = \frac{\pi}{3}$



Area $= \frac{1}{2}\times 4\times\frac{1}{4}\sin\frac{2\pi}{3}$

$= \frac{1}{4}\sqrt{3}$

(iii) $z_1 = r(\cos a + i\sin a)$, $z_2 = s(\cos b + i\sin b)$

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Area of triangle $= \frac{1}{2} rs \sin(b-a)$

$z_1^2 = r^2(\cos 2a + i \sin 2a)$, $z_2^2 = s^2(\cos 2b + i \sin 2b)$

Area of triangle $= \frac{1}{2} r^2 s^2 \sin(2b - 2a)$

$= \frac{1}{2} r^2 s^2 \sin 2(b-a)$

Areas are equal so $\frac{1}{2} rs \sin(b-a) = \frac{1}{2} r^2 s^2 \sin 2(b-a)$

$\frac{1}{2} rs \sin(b-a) = \frac{1}{2} r^2 s^2 \times 2 \sin(b-a) \cos(b-a)$

$1 = 2rs \cos(b-a)$

$\cos(b-a) = \dfrac{1}{2rs}$

(iv) If $s = \dfrac{1}{r}$

areas are equal $\Rightarrow \cos(b-a) = \dfrac{1}{2rs} = \dfrac{1}{2}$

$\Rightarrow b - a = \dfrac{\pi}{3}$ (since $0 \le a < b \le \frac{\pi}{2}$)

$b - a = \dfrac{\pi}{3} \Rightarrow b - a = \frac{1}{2}$

$\Rightarrow$ areas are equal

so areas are equal $\Leftrightarrow b - a = \dfrac{\pi}{3}$