Section 1: Modulus and argument

Solutions to Exercise level 2

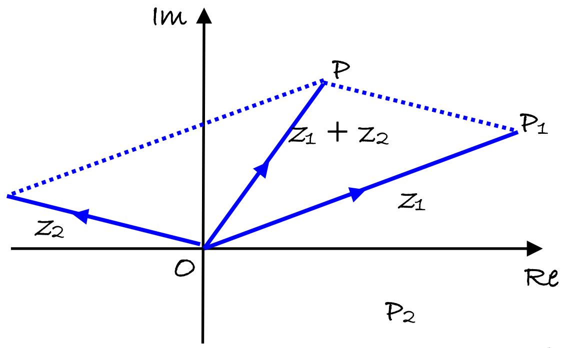
1. $z_1 = 12 + 5i \quad \Rightarrow |z_1| = \sqrt{12^2 + 5^2} = 13$

$z_2 = -3 + 4i \quad \Rightarrow |z_2| = \sqrt{3^2 + 4^2} = 5$

so $|z_1| + |z_2| = 18$

$z_1 + z_2 = 9 + 9i \quad \Rightarrow |z_1 + z_2| = \sqrt{9^2 + 9^2} = 9\sqrt{2} = 12.7$ (1 d.p.)

so $|z_1 + z_2| \leq |z_1| + |z_1|$.



The vector representing $z_1 + z_2$ is the diagonal of a parallelogram whose sides are formed by the vectors representing z_1 and z_2. So $|OP| = |z_1 + z_2|$

$|OP_1| = |P_2P| = |z_1|$

$|OP_2| = |P_1P| = |z_2|$

For any triangle, the length of one side is always shorter than the sum of the lengths of the other two sides.

For triangle OPP_1, $|OP| \leq |OP_1| + |P_1P|$

$|z_1 + z_2| \leq |z_1| + |z_2|$

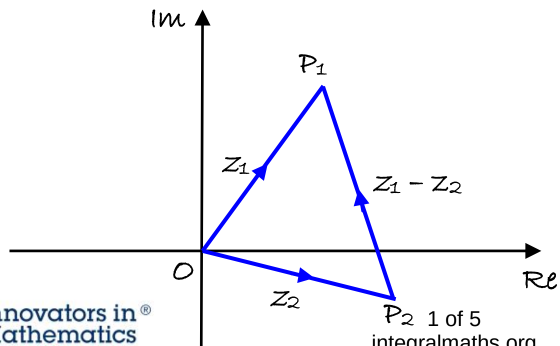
2. $z_1 = 12 + 5i \quad \Rightarrow |z_1| = \sqrt{12^2 + 5^2} = 13$

$z_2 = 3 - 4i \quad \Rightarrow |z_2| = \sqrt{3^2 + 4^2} = 5$

so $|z_1| - |z_2| = 8$

$z_1 - z_2 = 9 + 9i \quad \Rightarrow |z_1 - z_2| = \sqrt{9^2 + 9^2} = 9\sqrt{2} = 12.7$ (1 d.p.)

so $|z_1 - z_2| \geq |z_1| - |z_1|$.



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The vector representing $z_1 - z_2$ is the third sides of a triangle whose other two sides are formed by the vectors representing z_1 and z_2. So $|P_1P_2| = |z_1 - z_2|$

$|OP_1| = |z_1|$

$|OP_2| = |z_2|$

For any triangle, the length of one side is always shorter than the sum of the lengths of the other two sides.

For triangle OPP_1, $|OP_1| \le |OP_2| + |P_1P_2|$

$$|z_1| \le |z_2| + |z_1 - z_2|$$

$$|z_1 - z_2| \ge |z_1| - |z_2|$$

3. (i) $z = -2\sqrt{3} - 2i$ is in the third quadrant.

$$|z| = \sqrt{\left(2\sqrt{3}\right)^2 + 2^2} = \sqrt{12+4} = 4$$

$$\arg z = \arctan\left(\frac{2}{2\sqrt{3}}\right) - \pi = \arctan\left(\frac{1}{\sqrt{3}}\right) - \pi = \frac{\pi}{6} - \pi = -\frac{5\pi}{6}$$

$$z = 4\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$$

(ii) $$z = \frac{10}{\sqrt{3}-i} = \frac{10\left(\sqrt{3}+i\right)}{\left(\sqrt{3}-i\right)\left(\sqrt{3}+i\right)} = \frac{10\left(\sqrt{3}+i\right)}{4} = \frac{5}{2}\sqrt{3} + \frac{5}{2}i$$

This is in the first quadrant.

$$|z| = \tfrac{5}{2}\sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \tfrac{5}{2}\sqrt{4} = 5$$

$$\arg z = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

4. (i) $z = 1 + 2i$

$$|z| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

z is in the first quadrant, so $\arg z = \arctan\left(\frac{2}{1}\right) = 1.11$ (3 s.f.)

$$z = \sqrt{5}\left(\cos 1.11 + i\sin 1.11\right)$$

(ii) $z^* = 1 - 2i$

$$|z^*| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$\bar{z}$ is in the fourth quadrant, so $\arg z^* = \arctan\left(\frac{-2}{1}\right) = -1.11$ (3 s.f.)

$$z^* = \sqrt{5}\left(\cos(-1.11) + i\sin(-1.11)\right)$$

(iii) $\frac{1}{z} = \frac{1}{1+2i} = \frac{1-2i}{(1+2i)(1-2i)} = \frac{1-2i}{5}$

$\left|\frac{1}{z}\right| = \frac{1}{5}\sqrt{1^2+2^2} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$

$\frac{1}{z}$ is in the fourth quadrant, so $\arg\frac{1}{z} = \arctan\left(\frac{-2}{1}\right) = -1.11$ (3 s.f.)

$z = \frac{1}{\sqrt{5}}\left(\cos(-1.11) + i\sin(-1.11)\right)$

(iv) $\frac{1}{z^*} = \frac{1}{1-2i} = \frac{1+2i}{(1-2i)(1+2i)} = \frac{1+2i}{5}$

$\left|\frac{1}{z^*}\right| = \frac{1}{5}\sqrt{1^2+2^2} = \frac{\sqrt{5}}{5} = \frac{1}{\sqrt{5}}$

$\frac{1}{z^*}$ is in the first quadrant, so $\arg\frac{1}{z^*} = \arctan\left(\frac{2}{1}\right) = 1.11$ (3 s.f.)

$\frac{1}{z^*} = \frac{1}{\sqrt{5}}\left(\cos 1.11 + i\sin 1.11\right)$

$\left|\frac{1}{z}\right| = \left|\frac{1}{z^*}\right| = \frac{1}{|z|} = \frac{1}{|z^*|}$ and $\arg z = \arg\frac{1}{z^*} = -\arg z^* = -\arg\frac{1}{z}$.

5. (i) $w = 10i, \quad z = 1 + \sqrt{3}i$

$|w| = 10$

$\arg w = \frac{\pi}{2}$

$w = 10\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$|z| = \sqrt{1+3} = 2$

z is in the first quadrant so $\arg z = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$

$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(ii) $|wz| = |w||z| = 10 \times 2 = 20$

$\arg(wz) = \arg w + \arg z = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

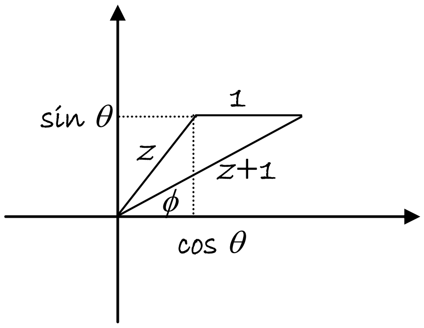
$wz = 20\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$\left|\frac{w}{z}\right| = \frac{|w|}{|z|} = \frac{10}{2} = 5$

$$\arg\left(\frac{w}{z}\right) = \arg w - \arg z = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

$$\frac{w}{z} = 5\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right)$$

6.



$$\tan\phi = \frac{\sin\theta}{\cos\theta + 1}$$

$$= \frac{2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta}{2\cos^2\tfrac{1}{2}\theta}$$

$$= \tan\tfrac{1}{2}\theta$$

$$\Rightarrow \phi = \tfrac{1}{2}\theta$$

$$\arg(z+1) = \tfrac{1}{2}\theta$$

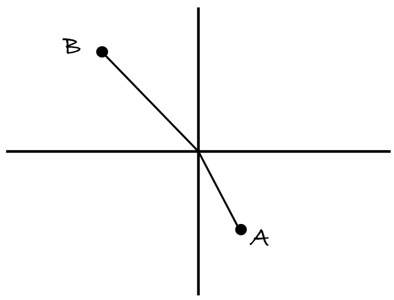
7. (i) $|\alpha| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$|\beta| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$

α is in the 4th quadrant, so $\arg\alpha = \arctan(-\sqrt{3}) = -\tfrac{\pi}{3}$

β is in the 2nd quadrant, so $\arg\beta = \arctan(-1) + \pi = -\tfrac{\pi}{4} + \pi = \tfrac{3\pi}{4}$

(ii)



(iii) $\left|\frac{\beta}{\alpha}\right| = \frac{|\beta|}{|\alpha|} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

$\arg\beta - \arg\alpha = \tfrac{3\pi}{4} - (-\tfrac{\pi}{3}) = \tfrac{13}{12}\pi$

Since $\tfrac{13}{12}\pi$ does not lie in the range $-\pi < \theta \le \pi$,

$$\arg\frac{\beta}{\alpha} = \tfrac{13}{12}\pi - 2\pi = -\tfrac{11}{12}\pi$$

$$\frac{\beta}{\alpha} = \sqrt{2}\left(\cos(-\tfrac{11}{12})\pi + i\sin(-\tfrac{11}{12})\pi\right)$$

(iv) The transformation is an enlargement by scale factor $\sqrt{2}$ and a rotation through $-\tfrac{11}{12}\pi$ about the origin (i.e. $\tfrac{11}{12}\pi$ clockwise).

8. (i) $z_1 = -1 + i$

$|z_1| = \sqrt{2}, \quad \arg z_1 = \pi - \arctan 1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z_2 = \sqrt{3} + i$

$|z_2| = 2, \quad \arg z_2 = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$$\left|\frac{z_1}{z_2}\right| = \frac{\sqrt{2}}{2}, \quad \arg\left(\frac{z_1}{z_2}\right) = \frac{3\pi}{4} - \frac{\pi}{6} = \frac{7\pi}{12}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

(ii) $$\frac{-1+i}{\sqrt{3}+i} = \frac{(-1+i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{1-\sqrt{3}+(1+\sqrt{3})i}{4} = \frac{1-\sqrt{3}}{4} + \frac{1+\sqrt{3}}{4}i$$

(iii) $$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2} \times \frac{2}{\sqrt{2}}\left(\frac{1-\sqrt{3}}{4} + \frac{1+\sqrt{3}}{4}i\right)$$

$$= \frac{\sqrt{2}}{2}\left(\frac{1-\sqrt{3}}{2\sqrt{2}} + \frac{1+\sqrt{3}}{2\sqrt{2}}i\right)$$

Comparing with result from (i) gives $\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}$

$\sin\frac{7\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$