Section 2: Finding the area under a curve

Solutions to Exercise level 2

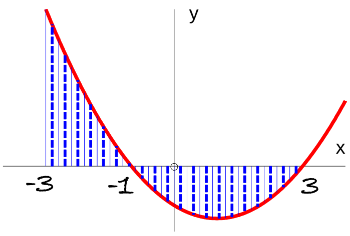
1. (i) $\int_{-2}^{2}(x+3)(x-2)\,dx = \int_{-2}^{2}(x^2+x-6)\,dx$

$$= \left[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 - 6x\right]_{-2}^{2}$$
$$= \left(\tfrac{8}{3}+2-12\right) - \left(-\tfrac{8}{3}+2+12\right)$$
$$= -\tfrac{56}{3}$$

(ii) $\int_{0}^{2} x(x^2+1)\,dx = \int_{0}^{3}\left(x^3+x\right)dx$

$$= \left[\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2\right]_{0}^{2}$$
$$= (4+2) - (0+0)$$
$$= 6$$

2. $y = x^2 - 2x - 3 = (x-3)(x+1)$



Area above x-axis $= \int_{-3}^{-1}(x^2-2x-3)\,dx$

$$= \left[\tfrac{1}{3}x^3 - x^2 - 3x\right]_{-3}^{-1}$$
$$= \left(-\tfrac{1}{3}-1+3\right) - \left(-9-9+9\right)$$
$$= \tfrac{32}{3}$$

Area below x-axis $= \left[\tfrac{1}{3}x^3 - x^2 - 3x\right]_{-1}^{3}$

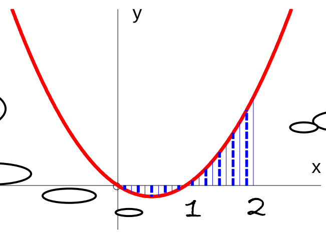
$$= \left(9-9-9\right) - \left(-\tfrac{1}{3}-1+3\right)$$
$$= -9 - \tfrac{5}{3}$$
$$= -\tfrac{32}{3}$$

Total area $= \tfrac{32}{3} + \tfrac{32}{3} = \tfrac{64}{3}$ square units

3. $y = x(x-1)$

The graph cuts the x-axis at the origin and the point (1, 0).

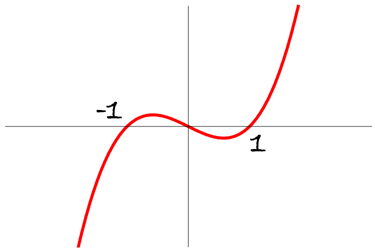
This area is negative as it is below the x-axis.



The areas above and below the x-axis must be calculated separately.

$$\text{Area below x-axis} = \int_0^1 x(x-1)\,dx$$
$$= \int_0^1 (x^2 - x)\,dx$$
$$= \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^1$$
$$= \tfrac{1}{3} - \tfrac{1}{2}$$
$$= -\tfrac{1}{6}$$
$$\text{Area above x-axis} = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_1^2$$
$$= \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - \tfrac{1}{2}\right)$$
$$= \tfrac{5}{6}$$
$$\text{Total area} = \tfrac{1}{6} + \tfrac{5}{6} = 1 \text{ square unit.}$$

4. (i) $y = x^3 - x = x(x^2 - 1) = x(x+1)(x-1)$

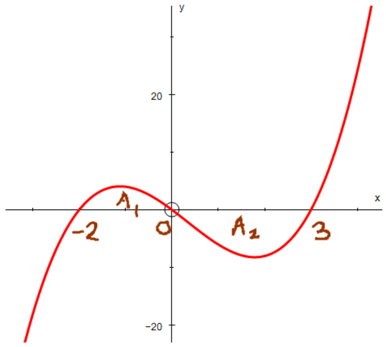


(ii) $\text{Area} = \int_1^2 (x^3 - x)\,dx$
$$= \left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_1^2$$
$$= (4-2) - \left(\tfrac{1}{4} - \tfrac{1}{2}\right)$$
$$= 2.25 \text{ square units}$$

(iii) $\text{Area} = \int_{-1}^0 (x^3 - x)\,dx$
$$= \left[\tfrac{1}{4}x^4 - \tfrac{1}{2}x^2\right]_{-1}^0$$
$$= 0 - \left(\tfrac{1}{4} - \tfrac{1}{2}\right)$$
$$= 0.25 \text{ square units}$$

(iv) By symmetry, the area between $x = 0$ and $x = 1$ is the same as the area between $x = -1$ and $x = 0$, only below the x-axis.
So the area between $x = 0$ and $x = 2$ is given by $2.25 + 0.25 = 2.5$ square units.

5. (i)



(ii) $A_1 = \int_{-2}^{0} x(x+2)(x-3)\,dx$

$= \int_{-2}^{0} x^3 - x^2 - 6x\,dx$

$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2\right]_{-2}^{0}$

$= 0 - \left(4 + \frac{8}{3} - 12\right)$

$= \frac{16}{3}$

$A_2 = \int_{0}^{3} x(x+2)(x-3)\,dx$

$= \left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2\right]_{0}^{3}$

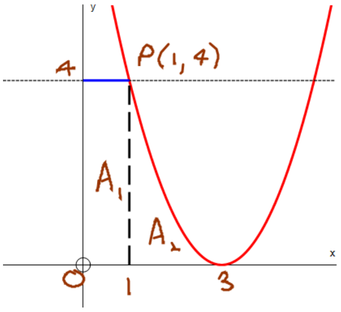
$= \left(\frac{81}{4} - \frac{27}{3} - 27\right) - (0)$

$= -\frac{63}{4}$

Total area $= \frac{16}{3} - (-\frac{63}{4}) = \frac{253}{12} \approx 21.1$ square units.

6. P is the point given by $4 = (x-3)^2 \Rightarrow x = 1$ or 5

so P = (1, 4)



Area = rectangle $+ \int_{1}^{3} (x-3)^2\,dx$

$= 4 + \int_{1}^{3} x^2 - 6x + 9\,dx$

$= 4 + \left[\frac{1}{3}x^3 - 3x^2 + 9x\right]_{1}^{3}$

$= 4 + (9 - 27 + 27) - (\frac{1}{3} - 3 + 9)$

$= \frac{20}{3}$ units^2

7. $y = x(4 - x^2) = x(2-x)(2+x)$

When $y = 0$, $x = 0$, $x = -2$, or $x = 2$

$$A_1 = \int_{-2}^{0} 4x - x^3 \, dx$$
$$= \left[2x^2 - \tfrac{1}{4}x^4\right]_{-2}^{0}$$
$$= 0 - (8 - 4) = -4$$
$$A_2 = \int_{0}^{2} 4x - x^3 \, dx$$
$$= \left[2x^2 - \tfrac{1}{4}x^4\right]_{0}^{2}$$
$$= (8 - 4) - 0 = 4$$
$$\text{Area} = 4 + 4 = 8$$