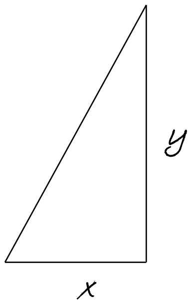
Section 4: More about differentiation

Solutions to Exercise level 3 (Extension)



1. $x + y = k \Rightarrow y = k - x$

   Area $A = \frac{1}{2}xy$

   $= \frac{1}{2}x(k-x)$

   $= \frac{1}{2}kx - \frac{1}{2}x^2$

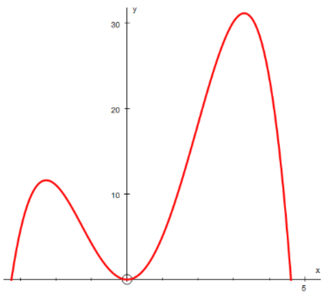
   $\Rightarrow \frac{dA}{dx} = \frac{1}{2}k - x$

   $\Rightarrow \frac{d^2A}{dx^2} = -1$

   $\frac{dA}{dx} = 0 \Rightarrow x = \frac{1}{2}k$, so $y = \frac{1}{2}k$

   (and $\frac{d^2A}{dx^2} < 0$, so $x = \frac{1}{2}k$ gives a maximum)

   So the triangle is isosceles, with area $= \frac{1}{8}k^2$.



2. (i) $V = \pi\left(\frac{1}{2}h\right)^2 h + \frac{1}{3}h^2 d$

   $= \frac{1}{4}\pi h^3 + \frac{1}{3}h^2 d$

   The required condition is $d + h = 15 - h^2$

   $\Rightarrow d = 15 - h - h^2$

   so $V = \frac{1}{4}\pi h^3 + 5h^2 - \frac{1}{3}h^3 - \frac{1}{3}h^4$

   $= 5h^2 + \left(\frac{1}{4}\pi - \frac{1}{3}\right)h^3 - \frac{1}{3}h^4$

   (ii) $\frac{dV}{dh} = 10h + 3\left(\frac{1}{4}\pi - \frac{1}{3}\right)h^2 - \frac{4}{3}h^3$

   $\frac{d^2V}{dh^2} = 10 + 6\left(\frac{1}{4}\pi - \frac{1}{3}\right)h - 4h^2$

   $\frac{dV}{dh} = 0 \Rightarrow h\left(10 + 3\left(\frac{1}{4}\pi - \frac{1}{3}\right)h - \frac{4}{3}h^2\right) = 0$

   $\Rightarrow h \approx -2.277$ or $h \approx 3.294$

   h cannot be negative, so $h \approx 3.294$ m

   $\Rightarrow d \approx 0.855$ m, $V \approx 31.17$ m^3

OCR AS Maths Differentiation 2 Exercise solutions

3. (i) $f'(x) = \lim_{h\to 0} \dfrac{(x+h)^2 - x^2}{h}$

$= \lim_{h\to 0} \dfrac{2xh + h^2}{h}$

$= \lim_{h\to 0} (2x + h)$

$= 2x$

(ii) $f'(x) = \lim_{h\to 0} \dfrac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$

$= \lim_{h\to 0} \dfrac{x^2 - (x+h)^2}{hx^2(x+h)^2}$

$= \lim_{h\to 0} \dfrac{-2xh - h^2}{hx^2(x+h)^2}$

$= \lim_{h\to 0} \dfrac{-2x - h}{x^2(x+h)^2}$

$= \dfrac{-2x}{x^2 x^2}$ since as $h \to 0, x + h \to x$

$= -\dfrac{2}{x^3}$

4. (i) Rate of filling is given by $R = (-t^2 + 10t) - (-\frac{1}{10}t^3 + \frac{6}{5}t^2)$

$= \frac{1}{10}t^3 - \frac{11}{5}t^2 + 10t$

$\dfrac{dR}{dt} = \frac{3}{10}t^2 - \frac{22}{5}t + 10$

For the maximum rate, $\dfrac{dR}{dt} = 0 \Rightarrow 3t^2 - 44t + 100 = 0$

$\Rightarrow t \approx 2.812$ or $t \approx 11.85$ (discard!)

When $t \approx 2.812$, $y \approx 12.95$

So the quickest rate is 12.95 litres / minute, after 2.812 minutes.

(ii) The tank stops filling when the 'in' and 'out' rates are equal.

$\Rightarrow -t^2 + 10x = -\frac{1}{10}t^3 + \frac{6}{5}t^2$

$\Rightarrow t^3 - 22t^2 + 100t = 0$

$\Rightarrow t\left(t^2 - 22t + 100\right) = 0$

$\Rightarrow t = 0$ (discard!) or $t \approx 6.417$ or $x \approx 15.58$ (discard!)

so the tank is at its fullest after 6.417 minutes.

(iii) When $t = 10$, $R = \frac{1}{10} \times 10^3 - \frac{11}{5} \times 10^2 + 10 \times 10 = -20$

After 10 minutes, the tank is emptying, at 20 litres/minute.