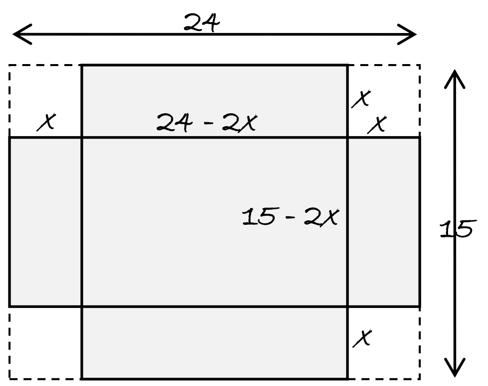
Section 4: More about differentiation

Solutions to Exercise level 2

1. Let the length of the sides be x and y.
   Considering the perimeter: $2(x+y)=20 \quad \Rightarrow x+y=10$
   Let the area be A: $A = xy$
   $= x(10-x)$
   $= 10x - x^2$
   $\frac{dA}{dx} = 10 - 2x$
   At turning point, $10 - 2x = 0$
   $2x = 10$
   $x = 5$
   When x = 5, y = 10 – 5 = 5.
   $\frac{d^2A}{dx^2} = -2$ so turning point is a maximum.
   The area is a maximum when the lengths of the sides are 5 cm
   (i.e. the rectangle is a square).

2. (i)



Height of box is x cm
Length of box is (24 – 2x) cm
Width of box is (15 – 2x) cm

Volume $V = x(15-2x)(24-2x)$
$= x(360 - 78x + 4x^2)$
$= 4x^3 - 78x^2 + 360x$

(ii) $\frac{dV}{dx} = 12x^2 - 156x + 360$

At turning points, $12x^2 - 156x + 360 = 0$

$$x^2 - 13x + 30 = 0$$
$$(x-3)(x-10) = 0$$
$$x = 3 \text{ or } x = 10$$

$x = 10$ is not possible since this would mean that the width would be negative.

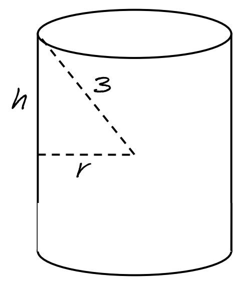
$$\frac{d^2V}{dx^2} = 24x - 156$$

When $x = 3$, $\frac{d^2V}{dx^2} = 72 - 156 < 0$, so $x = 1$ is a maximum point.

The volume of the box is maximised when $x = 3$.

(iii) Volume when $x = 3$ is $V = 3 \times 9 \times 18 = 486$ cm³

3. (i)



$$r^2 + h^2 = 3^2$$
$$r = \sqrt{9 - h^2}$$

(ii) Volume $V = \pi r^2 h$

$$= \pi(9 - h^2) \times 2h$$
$$= 2\pi h(9 - h^2)$$

(iii) $V = 18\pi h - 2\pi h^3$

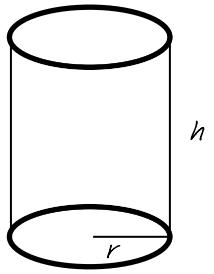
$$\frac{dV}{dh} = 18\pi - 6\pi h^2$$

At turning points, $18\pi - 6\pi h^2 = 0$

$$3 - h^2 = 0$$
$$h = \sqrt{3}$$

$\frac{d^2V}{dh^2} = -12\pi h$ so $h = \sqrt{3}$ is a maximum point.

Maximum volume $V = 2\pi\sqrt{3}(9-3) = 12\pi\sqrt{3}$ cm³.



4. $r + h = 24 \Rightarrow h = 24 - r$

$V = \pi r^2 h = \pi r^2(24 - r)$

$V = 24\pi r^2 - \pi r^3$

$\Rightarrow \frac{dV}{dr} = 48\pi r - 3\pi r^2$

$\Rightarrow \frac{d^2V}{dr^2} = 48\pi - 6\pi r$

$\frac{dV}{dr} = 0 \Rightarrow 3\pi r(16 - r) = 0$

$\Rightarrow r = 0$ (discard!) or $r = 16$

When $r = 16$, $\frac{d^2V}{dr^2} < 0$

so $r = 16$ gives the maximum volume, when $h = 8$

and volume $= 2048\pi$ m^3.

5. (i) Volume of a cylindrical can $= \pi r^2 \times h$

2 litres $= 0.002 m^3$ $(1 m^3 = 1000 litres)$

$0.002 = \pi r^2 h$

$h = \frac{0.002}{\pi r^2}$

(ii) Surface area $= 2\pi rh + 2\pi r^2$

$= 2\pi r(h + r)$

$= 2\pi r(\frac{0.002}{\pi r^2} + r)$

$= 2\pi(\frac{0.002}{\pi r} + r^2)$

(iii) $s = 2\pi(\frac{0.002}{\pi r} + r^2)$

$= \frac{0.004}{r} + 2\pi r^2 = 0.004r^{-1} + 2\pi r^2$

$\frac{ds}{dr} = -0.004r^{-2} + 4\pi r$

$\frac{ds}{dr} = 0 \Rightarrow -0.004r^{-2} + 4\pi r = 0$

$\Rightarrow 4\pi r = 0.004r^{-2}$

$\Rightarrow 4\pi r^3 = 0.004$

$\Rightarrow r^3 = \frac{0.004}{4\pi}$

$\Rightarrow r = 0.0683$ m (3 s.f.)

$\frac{d^2s}{dr^2} = 0.008r^{-3} + 4\pi$, which is positive for all values of r, so the stationary point must be a minimum point.

6. $y = x^3 + 2x^2$
When $x = -2$, $y = 0$
When $x = -2 + h$, $y = (h-2)^3 + 2(h-2)^2$
$= h^3 - 6h^2 + 12h - 8 + 2h^2 - 8h + 8$
$= h^3 - 4h^2 + 4h$
Gradient of chord $= \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{h^3 - 4h^2 + 4h - 0}{-2 + h - (-2)}$
$= \frac{h^3 - 4h^2 + 4h}{h}$
$= h^2 - 4h + 4$

7. (i) $y = 1 - x - x^3$
When $x = -1$, $y = 3$
When $x = -1 + h$, $y = 1 - (h-1) - (h-1)^3$
$= 1 - h + 1 - h^3 + 3h^2 - 3h + 1$
$= 3 - 4h + 3h^2 - h^3$
Gradient of chord $= \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{3 - 4h + 3h^2 - h^3 - 3}{-1 + h - (-1)}$
$= \frac{-h^3 + 3h^2 - 4h}{h}$
$= -h^2 + 3h - 4$

(ii) As $h \to 0$, gradient of chord $\to$ -4.
So the gradient of the tangent at P is -4.

8. (i) $f(x) = 2x^2 - 3x + 1$
$f(x+h) = 2(x+h)^2 - 3(x+h) + 1$
$= 2x^2 + 4xh + 2h^2 - 3x - 3h + 1$

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$$\text{Gradient of chord } = \frac{f(x+h)-f(x)}{(x+h)-x}\infty$$

$$= \frac{2x^2+4xh+2h^2-3x-3h+1-(2x^2-3x+1)}{x+h-x}$$

$$= \frac{4xh+2h^2-3h}{h}$$

$$= 4x+2h-3$$

As $h \to 0$, gradient of chord $\to 4x-3$.

So $f'(x) = 4x-3$.

(ii) $f(x) = x^3-2x^2+3$

$$f(x+h) = (x+h)^3-2(x+h)^2+3$$

$$= x^3+3x^2h+3xh^2+h^3-2x^2-4xh-2h^2+3$$

$$\text{Gradient of chord } = \frac{f(x+h)-f(x)}{(x+h)-x}$$

$$= \frac{x^3+3x^2h+3xh^2+h^3-2x^2-4xh-2h^2+3-(x^3-2x^2+3)}{x+h-x}$$

$$= \frac{3x^2h+3xh^2+h^3-4xh-2h^2}{h}$$

$$= 3x^2+3xh+h^2-4x-2h$$

As $h \to 0$, gradient of chord $\to 3x^2-4x$.

So $f'(x) = 3x^2-4x$.