Section 2: Maximum and minimum points

Solutions to Exercise level 3 (Extension)

1. (i) $x = 0,\ y = 3 \Rightarrow 3 = \frac{1}{3}k$

$\Rightarrow k = 9$

(ii) $\frac{dy}{dx} = \frac{1}{3}\left(-x^3 + 9x^2 - 23x + 15\right)$

so at $x = 0$, $\frac{dy}{dx} = 5$

and the ramp is given by $y = 5x + 3$.

(iii) At 10 metres horizontally, $x = 1$ and $\frac{dy}{dx} = 0$.

$x^3 - 9x^2 + 23x - 15 = 0$

$\Rightarrow (x-1)(x^2 - 8x + 15) = 0$

$\Rightarrow (x-1)(x-3)(x-5) = 0$

$\Rightarrow x = 1, \quad x = 3, \quad x = 5$

$y \approx 5.083, \quad y \approx 3.75, \quad y \approx 5.083$

so there are peaks at approximately (10, 50.8) and (50, 50.8) and a dip at approximately (30, 37.5), all measured in metres.

(iv) The track is steepest where the gradient is greatest, so this is where the rate of change of the gradient is zero.

Rate of change of gradient $= \frac{1}{3}\left(-3x^2 + 18x - 23\right)$

At steepest points $\Rightarrow 3x^2 - 18x + 23 = 0$

$\Rightarrow x \approx 4.154$ or $x \approx 1.845$

so gradient ≈ 1.026 or -1.026

For the steepest point, must also check the initial and final points:

$x = 0$ gives gradient 5

$x = 5$ gives gradient -5

So the track is steepest on the ramp (at the start) and again as it enters the tunnel after 50 metres horizontally, with the steepest points on the curved section at approximately 18.5 and 41.5 metres horizontally from the top of the ramp.

2. (i) $x = 0 \Rightarrow y = 6$

so the top of the ramp is 60 metres above the ground.

(ii) $x = 5 \Rightarrow y = 2$

so the vertical descent from the top of the ramp to the take-off point is 40 m.

(iii) $\frac{dy}{dx} = 0.09x^2 - 0.14x - 1.2$

$x = 5 \Rightarrow$ gradient of take-off point $= 0.35$

(iv) For the 'jump' equation

$\frac{dy}{dx} = -0.4x + 2.5$

so gradient of jump at take-off $= 0.5$

(v) The two gradients may not be equal because the jumper takes an 'upward leap' as he leaves the end of the ramp.

(iv) $-x + 9.8 = -0.2x^2 + 2.5x - 5.5 \Rightarrow 2x^2 - 35x + 153 = 0$

$\Rightarrow (x - 9)(2x - 17) = 0$

$\Rightarrow x = 8.5$ or $x = 9$

To land on the slope, the gradient of flight must be steeper than the gradient of the slope.

So checking, when $x = 8.5$, gradient of flight $= -0.9$

when $x = 9$, gradient of flight $= -1.1$

So he lands when $x = 9$.

(vii) The landing point is at (9, 0.8), the gradient of his flight at this point is -1.1, and the gradient of the slope at the landing point is -1.

(viii) The horizontal distance travelled in the jump = 9 – 5 = 4, so the jump travels 40 m horizontally.

(ix) The vertical drop from the top of the ramp to the landing $= 6 - 0.8 = 5.2$ so the real drop is 52 m vertically.

(x) For a soft landing, the directions of the flight and the landing slope must be similar, so the hill must be designed so that the gradients are similar.