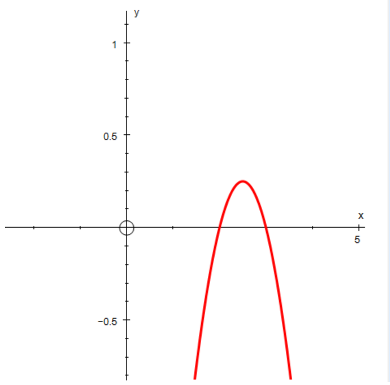
Section 1: Introduction to differentiation

Solutions to Exercise level 3 (Extension)



1. (i) When $y = -x^2 + 5x - 6$ crosses the x-axis,

$x^2 - 5x + 6 = 0$

$\Rightarrow (x-2)(x-3) = 0$

$\Rightarrow x = 2$ or $x = 3$

(ii) $\frac{dy}{dx} = -2x + 5$

so at P (2, 0), $\frac{dy}{dx} = 1$

and at Q (3, 0), $\frac{dy}{dx} = -1$

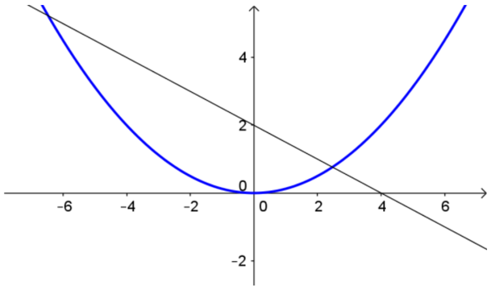
so the tangents at P and Q are perpendicular, and hence PABQ is a square.

So A = (2.5, 0.5) and B is (2.5, -0.5)

(iii) Area $= 2 \times$ area of triangle PAB

$= 2 \times \frac{1}{2} \times 1 \times 0.5$

$= 0.5$



2. (i) Equation of line through (0, 2) with gradient $-\frac{1}{2}$ is given by

$y = -\frac{1}{2}x + 2$

(ii) $2 - \frac{1}{2}x = \frac{1}{8}x^2 \Rightarrow x^2 + 4x - 16 = 0$

$\Rightarrow x = \frac{-4 \pm \sqrt{16+64}}{2}$

$\Rightarrow x = \frac{-4 \pm \sqrt{80}}{2} = \frac{-4 \pm 4\sqrt{5}}{2}$

so x-coordinates of P and Q are $-2 + 2\sqrt{5}$ and $-2 - 2\sqrt{5}$.

(iii) $\frac{dy}{dx} = \frac{1}{4}x$

so at P, gradient $= \frac{1}{2}(-1 + \sqrt{5})$

and at Q, gradient $= \frac{1}{2}(-1 - \sqrt{5})$

(iv) $m_P m_Q = \left[\frac{1}{2}(-1+\sqrt{5})\right]\left[\frac{1}{2}(-1-\sqrt{5})\right]$

$= \frac{1}{4}(1-5) = -1$

so tangents are perpendicular.

(v) $ax + 2 = \frac{1}{8}x^2 \Rightarrow x^2 - 8ax - 16 = 0$

$$\Rightarrow x = \frac{+8a \pm \sqrt{64a^2 + 64}}{2} = 4a \pm 4\sqrt{a^2+1}$$

so x-coordinates of R and S are $4a + 4\sqrt{a^2+1}$ and $4a - 4\sqrt{a^2+1}$.

(vi) $\frac{dy}{dx} = \frac{1}{4}x$

so at R, gradient is $a + \sqrt{a^2+1}$

and at S, gradient is $a - \sqrt{a^2+1}$

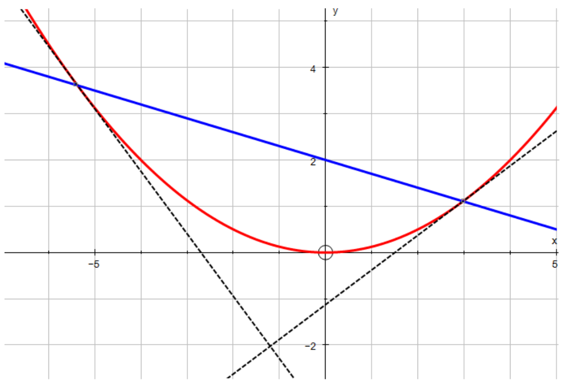
(vi)

$m_R m_S = \left(a + \sqrt{a^2+1}\right)\left(a - \sqrt{a^2+1}\right)$

$= a^2 - (a^2+1)$

$= -1$

so the tangents are perpendicular.



(viii) The equation $y = ax + 2$ is the general equation of a straight line through the point F (0, 2). Part (vii) shows by considering all values of a that for any straight line through F the tangents at the intersection points with the curve are always perpendicular.

{This is a particular example of a general result. The point F is the *focus* of this parabola, and perpendicularity is a general result for the tangents to the curve at the intersections with a line through the focus.}

3. (i) The graphs meet when $x^2 + x - 6 = 0$

$\Rightarrow (x-2)(x+3) = 0$

so P = (2, 2) and Q = (-3, 7)

At both of P and Q, the gradient of $y = -x + 4$ is -1.

$y = x^2 - 2 \Rightarrow \frac{dy}{dx} = 2x$

so at P, gradient = 4 and at Q, gradient = -6.

(ii) The crossing point with the larger x-coordinate is P (2, 2), and the gradients of the graph at P are -1 and 4.

So, from the diagram, the angle between the line and the tangent to the curve is given by

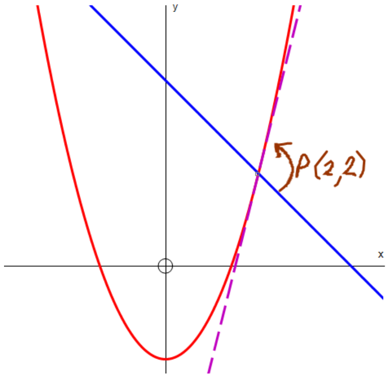
$\text{angle} = \alpha + \beta$

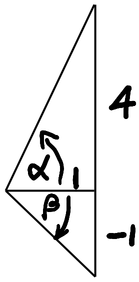
$= \tan^{-1}(\frac{4}{1}) + \tan^{-1}(\frac{-1}{1})$

$= 76.0^\circ + 45^\circ$

$= 121.0^\circ$ (to 3 s.f.)

So acute angle = 59.0° (3 s.f.)





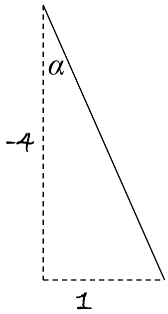
4. At A, $y = x^2 - 4x + 6$

$\Rightarrow \frac{dy}{dx} = 2x - 4$

so gradient = -4

$\Rightarrow \alpha = \tan^{-1} \frac{1}{4} = 14.1^\circ$ (3 s.f.)

By symmetry the angle at D is also 14.1° (3 s.f.)



At B, $x = 4$

The left-hand curve gives $\frac{dy}{dx} = 4 - 4 = 4$

For the right-hand curve, $y = \frac{1}{2}x^2 - 6x + 22$

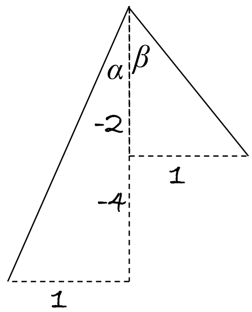
$\frac{dy}{dx} = x - 6$

so gradient = -2

$\beta = \tan^{-1} \frac{1}{2} = 26.6^\circ$ (3 s.f.)

so vertex B is $\alpha + \beta = 40.7^\circ$ (3 s.f.)

By symmetry vertex C is also 40.7° (3 s.f.).

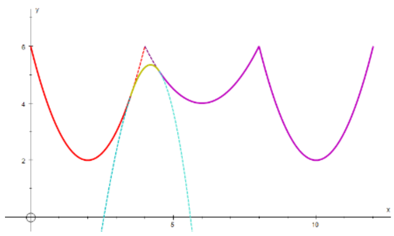


(ii) The new curve at point B is

$y = -0.25x^3 + 0.75x^2 + 6.938x - 18.5$

and it must meet the old curves at $x = 3.5$ and $x = 4.5$.

At $x = 3.5$, left curve gives $y = 4.25$



new curve gives $y = -0.25x^3 + 0.75x^2 + 6.938x - 18.5$

≈ 4.252

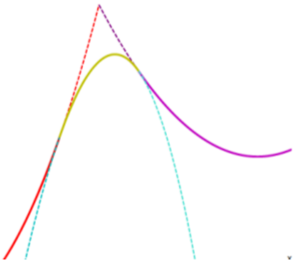
At $x = 4.5$, left curve gives $y = 5.125$

new curve gives

$y = -0.25x^3 + 0.75x^2 + 6.938x - 18.5$

≈ 5.127

so the new curve is a good fit to the original curves at $x = 3.5$ and $x = 4.5$.



(iii) For the curves at point B,

left-hand curve $y = (x-2)^2 + 2 = x^2 - 4x + 6$

$$\Rightarrow \frac{dy}{dx} = 2x - 4$$

When $x = 3.5$, gradient = 3

new curve $y = -0.25x^3 + 0.75x^2 + 6.938x - 18.5$

$$\Rightarrow \frac{dy}{dx} = -0.75x^2 + 1.5x + 6.938$$

When $x = 3.5$, gradient ≈ 3

right-hand curve $y = \frac{1}{2}x^2 - 6x + 22$

$$\Rightarrow \frac{dy}{dx} = x - 6$$

When $x = 4.5$, gradient = -1.5

new curve $y = -0.25x^3 + 0.75x^2 + 6.938x - 18.5$

$$\Rightarrow \frac{dy}{dx} = -0.75x^2 + 1.5x + 6.938$$

When $x = 4.5$, gradient ≈ -1.50

so the shape of the new curve fits the gradients of the old curves, so the computer-controlled machine can cut smoothly.