Section 2: Notation and proof

Solutions to Exercise level 3

1. A number greater than or equal to 100 can be written in the form
$100a + b$, where $a > 0$ and $0 \le b < 100$
$100a$ is divisible by 4, so b divisible by 4 $\Leftrightarrow 100a + b$ is divisible by 4

2. Let the roots be a and 2a.
So the equation can be written as $(x-a)(x-2a)=0$
$$\Rightarrow x^2 - 3ax + 2a^2 = 0$$
$$\Rightarrow p = 3a \text{ and } q = 2a^2$$
$$\Rightarrow 9q - 2p^2 = 9\times 2a^2 - 2\times(3a)^2 = 0$$
For the converse, $9q - 2p^2 = 0 \Rightarrow$ equation is $x^2 - px + \frac{2}{9}p^2 = 0$
$$\Rightarrow (x - \tfrac{1}{3}p)(x - \tfrac{2}{3}p) = 0$$
$$\Rightarrow \text{roots are } \tfrac{1}{3}p \text{ and } \tfrac{2}{3}p,$$
$\Rightarrow$ one root is twice the other provided $p \ne 0$.

3. 3 is prime
5 is prime
7 is prime
9 = 3x3 so is the product of two primes
11 is prime
13 is prime
15 = 3x5 so is the product of two primes
17 is prime
19 is prime
21 = 3x7 so is the product of two primes
23 is prime
25 = 5x5 so is the product of two primes
27 can only be written as 3x9 so it is neither a prime nor a product of two primes
so the statement is disproved

4. If the first digit is a and the second digit is b, then the last digit of b^2+a must be b.
If b = 1, b^2 = 1 so a can only be zero which means it is not a 2 digit number.
If b = 2, b^2 = 4 so a must be 8. 82 does not work as 8 + 2^2 is not 82
If b = 3, b^2 = 9 so a must be 4. 43 does not work as 4 + 3^2 is not 43
If b = 4, b^2 = 16 so a must be 8. 84 does not work as 8 + 4^2 is not 84
If b = 5, b^2 = 25 so a can only be zero which means it is not a 2 digit number.

If $b = 6$, $b^2 = 36$ so a can only be zero which means it is not a 2 digit number.
If $b = 7$, $b^2 = 49$ so a must be 8. 87 does not work as $8 + 7^2$ is not 87
If $b = 8$, $b^2 = 64$ so a must be 4. 48 does not work as $4 + 8^2$ is not 48
If $b = 9$, $b^2 = 81$ so a must be 8. 89 works as $8 + 9^2 = 89$

So the only number is 89.

5. (i) $[10b+c]^2 = 100b^2 + 20bc + c^2$

$$= 10[10b+2c]b + c^2$$

(ii) Any positive integer can be written as $10b + c$ for some integers b and c, and the expression for the square becomes $10k + c^2$ so the final digit of the square number is determined only by c^2.

For different values of c, we get

$1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 10 + 6$, $5^2 = 20 + 5$, $6^2 = 30 + 6$,
$7^2 = 40 + 9$, $8^2 = 60 + 4$, $9^2 = 80 + 1$, $0^2 = 0$

So squares end in 1, 4, 5, 6, or 9, and never in 2, 3, 7, or 8.