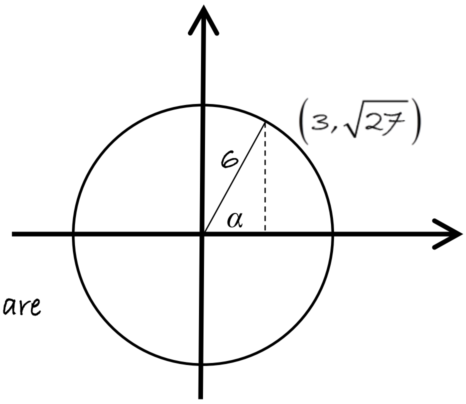
Section 2: Circles

Solutions to Exercise level 3 (Extension)



1. $3^2 + \left(\sqrt{27}\right)^2 = k^2 \Rightarrow k^2 = 36$

$$\Rightarrow k = 6$$

In the diagram, $\alpha = 60°$.
The three points are spaced out equally around the circle, so the angles at the centre are 120°.
So for an equilateral triangle the vertices Q and R are $\left(3, -\sqrt{27}\right)$ and (-6, 0).

2. (i) M is (6, 3), and the circle has radius $\sqrt{4^2 + 2^2} = \sqrt{20}$.
So the equation is $(x-6)^2 + (y-3)^2 = 20$

$$\Rightarrow x^2 + y^2 - 12x - 6y + 25 = 0$$

(ii) $\left.\begin{array}{r} y = 3x - 15 \\ x^2 + y^2 - 12x - 6y + 25 = 0 \end{array}\right\}$

$$\Rightarrow x^2 + (3x-15)^2 - 12x - 6(3x-15) + 25 = 0$$
$$\Rightarrow x^2 - 12x + 34 = 0$$
$$\Rightarrow x = \frac{12 \pm \sqrt{144 - 136}}{2}$$
$$= 6 \pm \sqrt{2}$$

So U is (7.41, 7.24) and V is (4.59, -1.24)
$\angle PUQ$ is an angle in a semicircle, so $\angle PUQ = 90°$

(iii) $\left.\begin{array}{r} y + 2x = 5 \\ x^2 + y^2 - 12x - 6y + 25 = 0 \end{array}\right\}$

$$\Rightarrow x^2 + (-2x+5)^2 - 12x - 6(-2x+5) + 25 = 0$$
$$\Rightarrow x^2 - 4x + 4 = 0$$
$$\Rightarrow (x-2)^2 = 0$$
$$\Rightarrow x = 2 \text{ (twice!)}, \ y = 1$$

So L_2 is a tangent to the circle at P, and therefore $\angle RPQ = 90°$.

3. For all the circles passing through P and Q, the line segment PQ is a chord. For each circle, the diameter through the midpoint of PQ is a perpendicular bisector. Therefore all centres lie along this line.

Midpoint of PQ is (3, 2)

Gradient of PQ $= \frac{10}{4} = \frac{5}{2}$

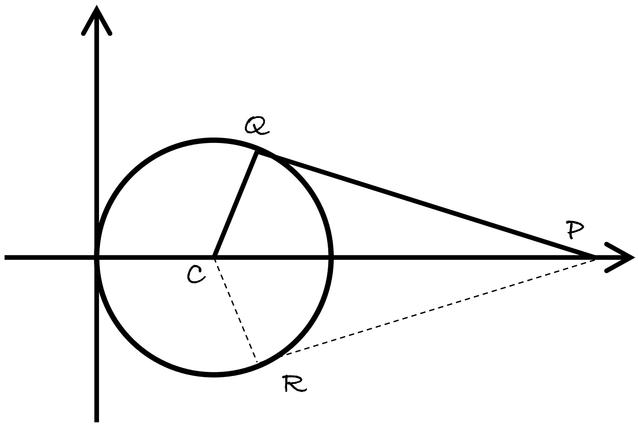
so the gradient of the perpendicular bisector $= -\frac{2}{5}$

So the equation of the line of centres is $y - 2 = -\frac{2}{5}(x - 3)$

$\Rightarrow 2x + 5y = 16$

4. (i) $(x-5)^2 + y^2 = 5^2$

$\Rightarrow x^2 + y^2 - 10x = 0$



(ii) $PQ = \sqrt{15^2 - 5^2}$

$= \sqrt{200} = 10\sqrt{2}$ m

(iii) Since Q = (a, b)

gradient $CQ = \dfrac{b}{a-5}$

gradient $QP = \dfrac{-b}{20-a}$

$\Rightarrow \dfrac{-b}{20-a} = -\dfrac{a-5}{b}$

$\Rightarrow -b^2 = (5-a)(20-a)$

$\Rightarrow -b^2 = 100 - 25a + a^2$

But Q lies on the circle, so $a^2 + b^2 - 10a = 0$

$\Rightarrow a^2 - 10a = 100 - 25a + a^2$

$\Rightarrow 15a = 100$

$\Rightarrow a = \frac{20}{3}$

$\Rightarrow b^2 = \frac{200}{3} - \frac{400}{9} = \frac{200}{9}$

$\Rightarrow b = \frac{10\sqrt{2}}{3}$

So $Q = \left(\frac{20}{3}, \frac{10\sqrt{2}}{3}\right)$ and $R = \left(\frac{20}{3}, -\frac{10\sqrt{2}}{3}\right)$

(iv) Shape PQCR is a kite.

Area $= \frac{1}{2}(QR)(CP)$

$= \frac{1}{2}\left(\frac{20\sqrt{2}}{3}\right)(15)$

$= 50\sqrt{2}$ m^2