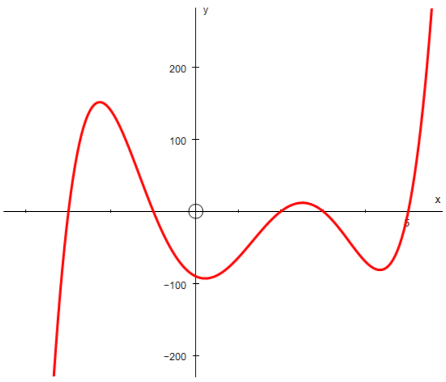
Section 1: Polynomial functions and graphs

Solutions to Exercise level 3 (Extension)

1. (i)



e.g. $y = (x+3)(x+1)(x-2)(x-3)(x-5)$ or many others with 5 or fewer points where it crosses the x-axis, and 4 local maxima/minima.

(ii) Putting $-x$ for x in $y = \frac{1}{120}x^5 - \frac{1}{6}x^3 + x$

gives $y = \frac{1}{120}(-x)^5 - \frac{1}{6}(-x)^3 + (-x)$

$= -\left(\frac{1}{120}x^5 - \frac{1}{6}x^3 + x\right)$

so the graph has half-turn symmetry about O (called an odd graph).

(iii) $\frac{1}{120}x^5 - \frac{1}{6}x^3 + x = 0$

$\Rightarrow \frac{1}{120}x(x^4 - 20x^2 + 120) = 0$

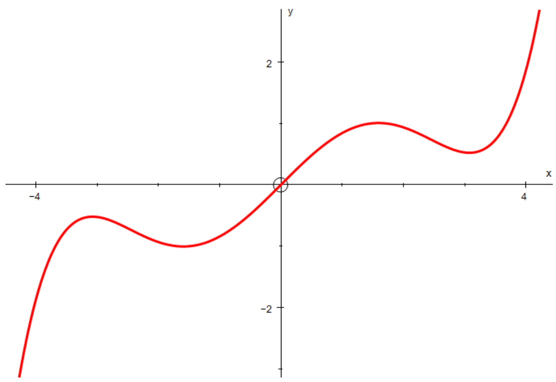
and for the quadratic expression in x^2, discriminant $= 20^2 - 4 \times 1 \times 120$

$= -80$

so there are no other intercepts other than $x = 0$.

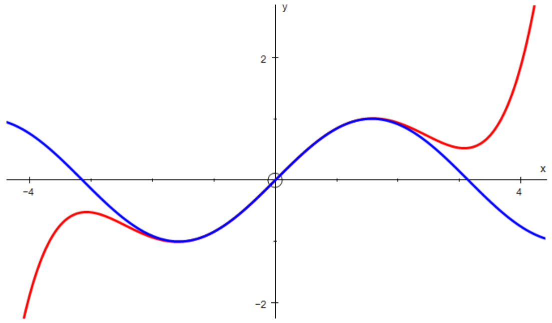
(iv) There is a maximum near (1.5, 1) and a minimum near (3 ,0.53), so from part (ii) there is a minimum near (-1.5, -1) and a maximum near (-3, -0.53).

(v)



(vi)

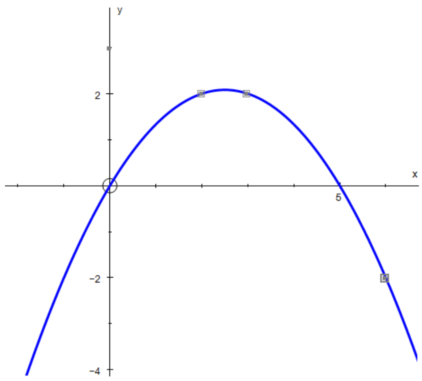
x	0	π/4	π/2	3π/4	π
$y = \sin x$	0	0.707	1	0.707	0
y = f(x)	0	0.707	1.005	0.781	0.524



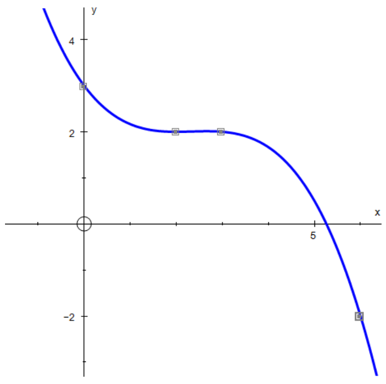
The polynomial could be a useful approximation for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

2. (i) point A gives $2 = 4a + 2b$
point B gives $2 = 9a + 3b$
$\Rightarrow a = -\frac{1}{3},\ b = \frac{5}{3}$

so Jane's graph is $y = -\frac{1}{3}x^2 + \frac{5}{3}x$

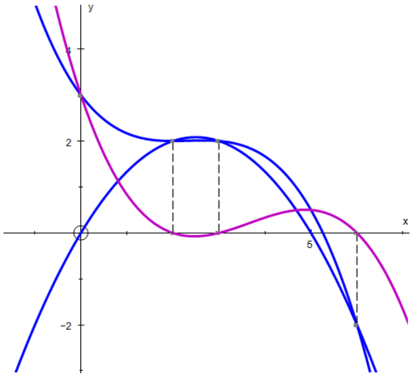


(ii) Samira's graph

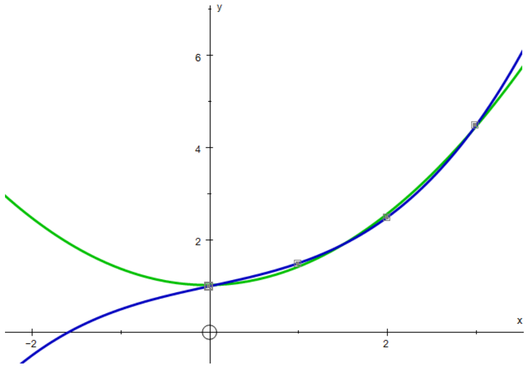


(iii) Mary's new polynomial is $y = -\frac{1}{12}x^3 + \frac{11}{12}x^2 - 3x + 3$

Since Jane's and Samira's graphs both pass through A, B and C, Mary's cubic polynomial has roots at $x = 2, 3,$ and 6.



3. (i)



(ii) The estimates are:

date	1600	1500
quadratic	1.375 m	2.475 m
cubic	0.5 m	-0.5 m

Both models give absurd estimates for 1500! And greatly different for 1600.

(iii) If $y = a(b + 2^x)$,

then $x = 0, y = 1 \Rightarrow 1 = a(b+1)$

and $x = 1, y = 1.5 \Rightarrow 1.5 = a(b+2)$

Dividing: $\dfrac{b+2}{b+1} = 1.5$

$\Rightarrow b = 1,\ a = \dfrac{1}{2}$

so $y = \dfrac{1}{2}(1 + 2^x)$ (and the other points fit exactly)

(iv) The predictions for year 2100 are:

quadratic polynomial gives 7.125 million

cubic polynomial gives 8.0 million

exponential function gives 8.5 million

For completeness, the calculations for all three models are below:

date	1500	1600	1700	1800	1900	2000	2100
x	-2	-1	0	1	2	3	4
y	??	??	1.0	1.5	2.5	4.5	??
quadratic	2.475	1.375	1.025	1.425	2.575	4.475	7.125
cubic	-0.5	0.5	1.0	1.5	2.5	4.5	8.0
exponential	0.625	0.75	1.0	1.5	2.5	4.5	8.5