Section 2: Numerical integration

Solutions to Exercise level 2

1. $f(x) = \dfrac{1}{\sqrt{1+\cos\theta}}$

   Using 4 strips, $h = \dfrac{\frac{\pi}{2} - 0}{3} = \dfrac{\pi}{6}$

   $y_0 = f(0) = \dfrac{1}{\sqrt{(1+\cos 0)}} = 0.70711$

   $y_1 = f\left(\dfrac{\pi}{6}\right) = \dfrac{1}{\sqrt{(1+\cos\frac{\pi}{6})}} = 0.73205$

   $y_2 = f\left(\dfrac{\pi}{3}\right) = \dfrac{1}{\sqrt{(1+\cos\frac{\pi}{3})}} = 0.81650$

   $y_3 = f\left(\dfrac{\pi}{2}\right) = \dfrac{1}{\sqrt{(1+\cos\frac{\pi}{2})}} = 1$

   Using the trapezium rule:

   $$\begin{aligned} \text{Area} &\approx \tfrac{1}{2}h\left[f_0 + f_3 + 2(f_1 + f_2)\right] \\ &= \tfrac{1}{2} \times \tfrac{\pi}{6}\left[0.70711 + 1 + 2(0.73205 + 0.81650)\right] \\ &= 1.258 \text{ (3 d.p.)} \end{aligned}$$

2. $f(x) = e^{-x^2}$

   (i) Using 2 strips, $h = \dfrac{1-0}{2} = \dfrac{1}{2}$

   $y_0 = f(0) = e^0 = 1$

   $y_1 = f(0.5) = e^{-0.5^2} = 0.77880$

   $y_2 = f(1) = e^{-1^2} = 0.36788$

   Using the trapezium rule:

   $$\begin{aligned} \int_0^1 e^{-x^2}\,dx &\approx \tfrac{1}{2}h\left(y_0 + y_2 + 2y_1\right) \\ &= \tfrac{1}{2} \times 0.5\left(1 + 0.36788 + 2 \times 0.77880\right) \\ &= 0.731 \text{ (3 d.p.)} \end{aligned}$$

   (ii) Using 4 strips, $h = \dfrac{1-0}{4} = \dfrac{1}{4}$

   $y_0 = f(0) = e^0 = 1$

   $y_1 = f(0.25) = e^{-0.25^2} = 0.93941$

   $y_2 = f(0.5) = e^{-0.5^2} = 0.77880$

   $y_3 = f(0.75) = e^{-0.75^2} = 0.56978$

   $y_4 = f(1) = e^{-1^2} = 0.36788$

Using the trapezium rule:

$$\int_0^1 e^{-x^2}dx \approx \tfrac{1}{2}h\left(y_0 + y_4 + 2(y_1 + y_2 + y_3)\right)$$
$$= \tfrac{1}{2}\times 0.25\left(1 + 0.36788 + 2(0.93941 + 0.77880 + 0.56978)\right)$$
$$= 0.743 \quad (3 \text{ d.p.})$$

$$y = e^{-x^2}$$
$$\frac{dy}{dx} = -2xe^{-x^2}$$
$$\frac{d^2y}{dx^2} = -2e^{-x^2} - 2x \times -2xe^{-x^2} = (4x^2 - 2)e^{-x^2}$$

$\frac{d^2y}{dx^2} = 0$ when $x = \frac{1}{\sqrt{2}}$ so at this point the curve changes from being concave to convex. So before that point the trapezia give underestimates and after that point the trapezia give overestimates.

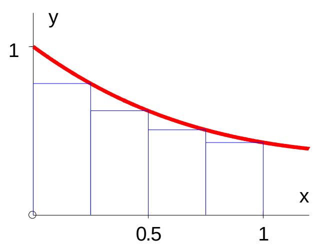
3. (i) $f(x) = e^{-\sin x}$

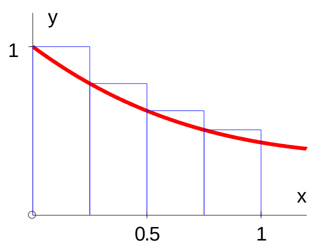
$$f'(x) = -\cos x\, e^{-\sin x}$$
$$f''(x) = \sin x\, e^{-\sin x} - \cos x \times -\cos x\, e^{-\sin x}$$
$$= (\sin x + \cos^2 x)\, e^{-\sin x}$$

Since $\sin x$ is positive for all values of x between 0 and 1, and $\cos^2 x$ is always positive, and $e^{-\sin x}$ is always positive, the value of the second derivative is positive over the region of the integral, so the curve is convex and hence the trapezium rule will give an overestimate.

(ii) $\int_0^1 e^{-\sin x}$

$f(x) = e^{-\sin x}$





(a) Underestimate $= 0.25\left(f(0.25) + f(0.5) + f(0.75) + f(1)\right)$

$$= 0.25\left(e^{-\sin 0.25} + e^{-\sin 0.5} + e^{-\sin 0.75} + e^{-\sin 1}\right)$$
$$= 0.5842$$

$$\text{Overestimate} = 0.25\left(f(0)+f(0.25)+f(0.5)+f(0.75)\right)$$
$$= 0.25\left(e^{-\sin 0}+e^{-\sin 0.25}+e^{-\sin 0.5}+e^{-\sin 0.75}\right)$$
$$= 0.7264$$

(b) $$\text{Underestimate} = 0.1\left(f(0.1)+f(0.2)+f(0.3)+\ldots+f(1)\right)$$
$$= 0.6235$$

$$\text{Overestimate} = 0.1\left(f(0)+f(0.1)+f(0.2)+\ldots+f(0.9)\right)$$
$$= 0.6804$$

4\. (i) $f(x) = \left(1+x^2\right)^{12}$

$h = 0.1$

$f_0 = f(0) = 1$

$f_1 = f(0.1) = 1.01^{12}$

$f_2 = f(0.2) = 1.04^{12}$

$f_3 = f(0.3) = 1.09^{12}$

$f_4 = f(0.4) = 1.16^{12}$

$f_5 = f(0.5) = 1.25^{12}$

Using the trapezium rule:

$$\text{Area} \approx \tfrac{1}{2}h\left[f_0+f_5+2\left(f_1+f_2+f_3+f_4\right)\right]$$
$$= \tfrac{1}{2}\times 0.1\left[1+1.25^{12}+2\left(1.01^{12}+1.04^{12}+1.09^{12}+1.16^{12}\right)\right]$$
$$= 1.9253 \text{ (4 d.p.)}$$

(ii) $$\left(1+x^2\right)^{12} = 1+12x^2+\frac{12\times 11}{1\times 2}(x^2)^2+\frac{12\times 11\times 10}{1\times 2\times 3}(x^2)^3 + \frac{12\times 11\times 10\times 9}{1\times 2\times 3\times 4}(x^2)^4+\ldots$$
$$= 1+12x^2+66x^4+220x^6+495x^8+\ldots.$$

(iii) $$\int_0^{0.5}\left(1+x^2\right)^{12}dx \approx \int_0^{0.5}\left(1+12x^2+66x^4+220x^6+495x^8\right)dx$$
$$= \left[x+4x^3+\tfrac{66}{5}x^5+\tfrac{220}{7}x^7+55x^9\right]_0^{0.5}$$
$$= 1.7655 \text{ (4 d.p.)}$$

(iv) The curve is convex so the trapezia will lie above the curve and therefore the trapezium rule will be an overestimate.
All the terms in the binomial expansion are positive and so using only some of the terms will result in an underestimate.
The estimate in (i) could be improved by using more strips.

The estimate in (iii) could be improved by using more terms in the expansion.

5. (i) $f(x) = \ln(x^2 + 1)$

$f(0) = \ln 1 = 0$

$f(0.2) = \ln 1.04$

$f(0.4) = \ln 1.16$

$f(0.6) = \ln 1.36$

$f(0.8) = \ln 1.64$

$f(1) = \ln 2$

$\text{Area} \approx \tfrac{1}{2} \times 0.2\left[0 + \ln 2 + 2\left(\ln 1.04 + \ln 1.16 + \ln 1.36 + \ln 1.64\right)\right]$

$= 0.2673$ (4 s.f.)

(ii) $f(x) = \ln(x^2 + 1)$

$$f'(x) = \frac{2x}{x^2 + 1}$$

$$f''(x) = \frac{2(x^2 + 1) - 2x(2x)}{(x^2 + 1)^2} = \frac{2 - 2x^2}{(x^2 + 1)^2}$$

$f''(x) \geq 0$ for $0 \leq x \leq 1$, so the trapezia are above the curve and therefore the estimate is an overestimate.

(iii) Using rectangles to find a lower bound:

$\text{Area} \approx 0.2\left[0 + \ln 1.04 + \ln 1.16 + \ln 1.36 + \ln 1.64\right]$

$= 0.1980$ (4 s.f.)