Section 4: Integration by parts

Solutions to Exercise level 3

1. (i) $u = \cos x \quad \Rightarrow \dfrac{du}{dx} = -\sin x$

$\dfrac{dv}{dx} = \cos x \quad \Rightarrow v = \sin x$

$$\int \cos^2 x\,dx = \cos x \sin x - \int -\sin^2 x\,dx$$
$$= \cos x \sin x + \int (1 - \cos^2 x)\,dx$$
$$= \cos x \sin x + x - \int \cos^2 x\,dx$$

$$2\int \cos^2 x\,dx = \cos x \sin x + x + k$$
$$\int \cos^2 x\,dx = \tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}x + c$$

(ii) $\int \cos^4 x\,dx = \int \cos x \cos^3 x\,dx$

$u = \cos^3 x \quad \Rightarrow \dfrac{du}{dx} = -3\cos^2 x \sin x$

$\dfrac{dv}{dx} = \cos x \quad \Rightarrow v = \sin x$

$$\int \cos^4 x\,dx = \cos^3 x \sin x - \int -3\cos^2 x \sin x \sin x$$
$$= \cos^3 x \sin x + \int 3\cos^2 x(1 - \cos^2 x)dx$$
$$\int \cos^4 x\,dx = \cos^3 x \sin x + \int 3\cos^2 x - 3\int \cos^4 x\,dx$$
$$4\int \cos^4 x\,dx = \cos^3 x \sin x + \int 3\cos^2 x$$
$$= \cos^3 x \sin x + \tfrac{3}{2}\cos x \sin x + \tfrac{3}{2}x + k$$
$$\int \cos^4 x\,dx = \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{8}\cos x \sin x + \tfrac{3}{8}x + c$$

2. (i) $u = \sin bx \quad \Rightarrow \dfrac{du}{dx} = b\cos bx$

$\dfrac{dv}{dx} = e^{ax} \quad \Rightarrow v = \dfrac{1}{a}e^{ax}$

$$I = \int e^{ax} \sin bx\,dx = \frac{1}{a}e^{ax}\sin bx - \int \frac{1}{a}e^{ax} b\cos bx\,dx$$
$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx$$

For $\int e^{ax}\cos bx\,dx$

$$u = \cos bx \quad \Rightarrow \frac{du}{dx} = -b\sin bx$$

$$\frac{dv}{dx} = e^{ax} \quad \Rightarrow v = \frac{1}{a}e^{ax}$$

$$I = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\left(\frac{1}{a}e^{ax}\cos bx - \int -\frac{1}{a}e^{ax}b\sin bx\,dx\right)$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \int \frac{b^2}{a^2}e^{ax}\sin bx\,dx$$

$$= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}I$$

$$\left(1 + \frac{b^2}{a^2}\right)I = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}e^{ax}\cos bx$$

$$\left(\frac{a^2+b^2}{a^2}\right)I = \frac{1}{a^2}e^{ax}\left(a\sin bx - b\cos bx\right)$$

$$I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

(ii) $$\int_0^\infty e^{-2x}\sin 3x\,dx = \left[\frac{e^{-2x}}{(-2)^2+3^2}(-2\sin 3x - 3\cos 3x)\right]_0^\infty$$

$$= \left[\frac{e^{-2x}}{13}(-2\sin 3x - 3\cos 3x)\right]_0^\infty$$

As $x \to \infty$, $e^{-2x} \to 0$ and $-2\sin 3x - 3\cos 3x$ is bounded, so the whole expression $\to 0$

$$\left[\frac{e^{-2x}}{13}(-2\sin 3x - 3\cos 3x)\right]_0^\infty = 0 - \frac{1}{13}(0-3)$$

$$= \frac{3}{13}$$

3. $$\int e^{\sqrt{x}}dx = \int x^{\frac{1}{2}}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}dx$$

Let $z = x^{\frac{1}{2}} \quad \Rightarrow \frac{dz}{dx} = \frac{1}{2}x^{-\frac{1}{2}} \quad \Rightarrow dx = 2x^{\frac{1}{2}}dz$

$$\int e^{\sqrt{x}}dx = \int zx^{-\frac{1}{2}}e^{z} \times 2x^{\frac{1}{2}}dz$$

$$= \int 2ze^{z}dz$$

Let $u = z \quad \Rightarrow \frac{du}{dz} = 1$

Let $\frac{dv}{dz} = e^{z} \quad \Rightarrow v = e^{z}$

$$2\int ze^z dz = 2ze^z - 2\int e^z dz$$
$$= 2ze^z - 2e^z + c$$
$$\int e^{\sqrt{x}} dx = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$$

