Section 4: Integration by parts

Solutions to Exercise level 2

1. (i) Let $u = x \Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = (1+x)^{-\frac{1}{2}} \Rightarrow v = 2(1+x)^{\frac{1}{2}}$

$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$

$$\int_0^3 \frac{x}{\sqrt{1+x}}dx = \left[2x(1+x)^{\frac{1}{2}}\right]_0^3 - \int_0^3 2(1+x)^{\frac{1}{2}}dx$$

$$= \left[2x(1+x)^{\frac{1}{2}} - \tfrac{4}{3}(1+x)^{\frac{3}{2}}\right]_0^3$$

$$= 6\sqrt{4} - \tfrac{4}{3}(4)^{\frac{3}{2}} + \tfrac{4}{3}\times 1$$

$$= 12 - \tfrac{32}{3} + \tfrac{4}{3}$$

$$= \tfrac{8}{3}$$

(ii) Let $u = 1 + x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

When $x = 0,\ u = 1$

When $x = 3,\ u = 4$

$$\int_0^3 \frac{x}{\sqrt{1+x}}dx = \int_1^4 \frac{u-1}{\sqrt{u}}du$$

$$= \int_1^4 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du$$

$$= \left[\tfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4$$

$$= \left(\tfrac{2}{3}\times 8 - 2\times 2\right) - \left(\tfrac{2}{3} - 2\right)$$

$$= \tfrac{8}{3}$$

2. (i) Let $u = \ln 2x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

Let $\frac{dv}{dx} = x^3 \Rightarrow v = \frac{1}{4}x^4$

Using definite integration by parts, $\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$

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$$\begin{aligned}\int_1^2 x^3 \ln 2x \, dx &= \left[\tfrac{1}{4}x^4 \ln 2x\right]_1^2 - \int_1^2 \tfrac{1}{4}x^4 \times \frac{1}{x} dx \\ &= \left[\tfrac{1}{4}x^4 \ln 2x\right]_1^2 - \int_1^2 \tfrac{1}{4}x^3 dx \\ &= \left[\tfrac{1}{4}x^4 \ln 2x - \tfrac{1}{16}x^4\right]_1^2 \\ &= (4\ln 4 - 1) - (\tfrac{1}{4}\ln 2 - \tfrac{1}{16}) \\ &= 8\ln 2 - 1 - \tfrac{1}{4}\ln 2 + \tfrac{1}{16} \\ &= \tfrac{31}{4}\ln 2 - \tfrac{15}{16}\end{aligned}$$

(ii) Let $u = \ln x \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$

Let $\dfrac{dv}{dx} = 1 \Rightarrow v = x$

Using definite integration by parts, $\int_a^b u\dfrac{dv}{dx}dx = [uv]_a^b - \int_a^b v\dfrac{du}{dx}dx$

$$\begin{aligned}\int_1^2 \ln x \, dx &= [x\ln x]_1^2 - \int_1^2 \frac{1}{x} \times x \, dx \\ &= [x\ln x]_1^2 - \int_1^2 1 \, dx \\ &= [x\ln x - x]_1^2 \\ &= (2\ln 2 - 2) - (\ln 1 - 1) \\ &= 2\ln 2 - 1\end{aligned}$$

3. $$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\ln|\cos x| + c\end{aligned}$$

Let $u = x \Rightarrow \dfrac{du}{dx} = 1$

Let $\dfrac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\begin{aligned}\int x\sec^2 x dx &= x\tan x - \int \tan x \, dx \\ &= x\tan x + \ln|\cos x| + c\end{aligned}$$

4. Area $= \int_0^1 xe^{-x} dx$

Let $u = x \Rightarrow \dfrac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$

$$\begin{aligned}\text{Area } = \int_0^1 xe^{-x}dx &= \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}dx \\ &= \left[-xe^{-x} - e^{-x}\right]_0^1 \\ &= \left(-e^{-1} - e^{-1}\right) - (-1) \\ &= -2e^{-1} + 1 \\ &= 1 - \frac{2}{e}\end{aligned}$$

5. Area $= \int_0^{\frac{\pi}{2}} x\sin 2x\,dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \sin 2x \Rightarrow v = -\tfrac{1}{2}\cos 2x$

$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$

$$\begin{aligned}\text{Area } = \int_0^{\frac{\pi}{2}} x\sin 2x\,dx &= \left[-\tfrac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\tfrac{1}{2}\cos 2x\,dx \\ &= \left[-\tfrac{1}{2}x\cos 2x + \tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} \\ &= \left(-\tfrac{\pi}{4}\cos\pi + \tfrac{1}{4}\sin\pi\right) - \left(0 + \tfrac{1}{4}\sin 0\right) \\ &= \tfrac{\pi}{4}\end{aligned}$$

6. (i) Let $u = x \Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^{3x} \Rightarrow v = \tfrac{1}{3}e^{3x}$

Using integration by parts, $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

$$\begin{aligned}\int xe^{3x}dx &= x \times \tfrac{1}{3}e^{3x} - \int 1 \times \tfrac{1}{3}e^{3x}\,dx \\ &= \tfrac{1}{3}xe^{3x} - \int \tfrac{1}{3}e^{3x}\,dx \\ &= \tfrac{1}{3}xe^{3x} - \tfrac{1}{3} \times \tfrac{1}{3}e^{3x} + c \\ &= \tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x} + c\end{aligned}$$

(ii) The derivative of e^{x^3} is $3x^2e^{x^3}$

By inspection, $\int x^2e^{x^3}dx = \tfrac{1}{3}\int 3x^2e^{x^3}dx$

$= \tfrac{1}{3}e^{x^3} + c$

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(iii) Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

Let $\frac{dv}{dx} = e^{3x} \Rightarrow v = \frac{1}{3}e^{3x}$

Using integration by parts, $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

$$\int x^2e^{3x}dx = x^2 \times \tfrac{1}{3}e^{3x} - \int 2x \times \tfrac{1}{3}e^{3x}dx$$
$$= \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\int xe^{3x}dx$$

Now integrate $\int xe^{3x}dx$ using integration by parts.

Let $u = x \Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^{3x} \Rightarrow v = \frac{1}{3}e^{3x}$

Using integration by parts, $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

$$\int xe^{3x}dx = x \times \tfrac{1}{3}e^{3x} - \int 1 \times \tfrac{1}{3}e^{3x}\,dx$$
$$= \tfrac{1}{3}xe^{3x} - \tfrac{1}{3}\int e^{3x}\,dx$$
$$= \tfrac{1}{3}xe^{3x} - \tfrac{1}{3} \times \tfrac{1}{3}e^{3x} + c$$
$$= \tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x} + c$$

$$\int x^2e^{3x}dx = \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{3}\left(\tfrac{1}{3}xe^{3x} - \tfrac{1}{9}e^{3x}\right) + c$$
$$= \tfrac{1}{3}x^2e^{3x} - \tfrac{2}{9}xe^{3x} + \tfrac{2}{27}e^{3x} + c$$