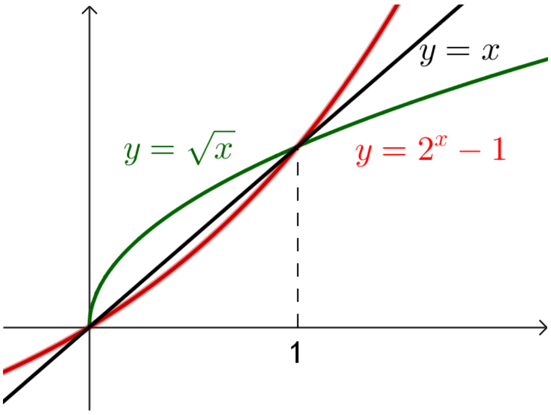
Section 2: Integration by substitution

Solutions to Exercise level 3

1.



So starting from smallest: $\int_0^1 (2^x - 1)\, dx$, $\int_0^1 x\, dx$, $\int_0^1 \sqrt{x}\, dx$

2. (i) $\sin x = A(-\sin x + \cos x) + B(\cos x + \sin x)$

$= (-A + B)\sin x + (A + B)\cos x$

Equating coefficients of $\cos x \quad \Rightarrow A + B = 0$

Equating coefficients of $\sin x \quad \Rightarrow -A + B = 1$

so $A = -\frac{1}{2}, B = \frac{1}{2}$

(ii)
$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\cos x + \sin x} dx = \int_0^{\frac{\pi}{2}} \frac{-\frac{1}{2}(-\sin x + \cos x) + \frac{1}{2}(\sin x + \cos x)}{\cos x + \sin x} dx$$
$$= \int_0^{\frac{\pi}{2}} \left(-\frac{1}{2}\left(\frac{-\sin x + \cos x}{\cos x + \sin x} \right) + \frac{1}{2} \right) dx$$
$$= \left[-\tfrac{1}{2}\ln|\cos x + \sin x| + \tfrac{1}{2}x \right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}\ln 1 + \tfrac{1}{4}\pi + \tfrac{1}{2}\ln 1 - 0$$
$$= \frac{\pi}{4}$$

3. For $n = 1$: $\int_0^{\pi/4} (\sin x + \cos x) dx = [-\cos x + \sin x]_0^{\pi/4}$
$$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (-1 + 0)$$
$$= 1$$

For $n=2$: $\int_0^{\pi/4}(\sin^2 x+\cos^2 x)dx=\int_0^{\pi/4}1\,dx=$

$$=[x]_0^{\pi/4}$$

$$=\frac{\pi}{4}$$

For $n=3$: $\int_0^{\pi/4}(\sin^3 x+\cos^3 x)dx$

$$=\int_0^{\pi/4}(\sin x(1-\cos^2 x)+\cos x(1-\sin^2 x)dx$$

$$=\int_0^{\pi/4}(\sin x-\sin x\cos^2 x+\cos x-\cos x\sin^2 x)dx$$

$$=\left[-\cos x+\tfrac{1}{3}\cos^3 x+\sin x-\tfrac{1}{3}\sin^3 x\right]_0^{\pi/4}$$

$$=-\frac{1}{\sqrt{2}}+\frac{1}{3(\sqrt{2})^3}+\frac{1}{\sqrt{2}}-\frac{1}{3(\sqrt{2})^3}-\left(-1+\frac{1}{3}+0\right)$$

$$=\frac{2}{3}$$

For $n=4$: $\int_0^{\pi/4}(\sin^4 x+\cos^4 x)dx$

$$=\int_0^{\pi/4}(\tfrac{1}{4}(1-\cos 2x)^2+\tfrac{1}{4}(1+\cos 2x)^2)dx=$$

$$=\tfrac{1}{4}\int_0^{\pi/4}(1-2\cos 2x+\cos^2 2x+1+2\cos 2x+\cos^2 2x)dx$$

$$=\tfrac{1}{2}\int_0^{\pi/4}(1+\cos^2 2x)dx$$

$$=\tfrac{1}{2}\int_0^{\pi/4}(1+\tfrac{1}{2}(1+\cos 4x)dx$$

$$=\tfrac{1}{2}\left[\tfrac{3}{2}x+\tfrac{1}{8}\sin 4x\right]_0^{\pi/4}$$

$$=\tfrac{3}{4}\times\tfrac{\pi}{4}+0$$

$$=\frac{3}{16}\pi$$

$$\sum_1^4\left(\int_0^{\pi/4}(\sin^n x+\cos^n x)dx\right)=1+\frac{\pi}{4}+\frac{2}{3}+\frac{3}{16}\pi$$

$$=\frac{5}{3}+\frac{7}{16}\pi$$

4. (i) The graph is symmetric about the line $x=\tfrac{1}{2}\pi$

(ii) $y=\pi-x \quad \Rightarrow \dfrac{dy}{dx}=-1 \quad \Rightarrow dx=-dy$

When $x=0, y=\pi$

When $x=\pi, y=0$

Edexcel A level Maths Integration 2 Exercise solutions

$$I = \int_0^{\pi} x f(x)\,dx = \int_{\pi}^{0} (\pi - y) f(\pi - y) \times -1\,dy$$
$$= -\int_{\pi}^{0} (\pi - y) f(y)\,dy$$
$$= \int_0^{\pi} (\pi - y) f(y)\,dy$$
$$= \pi \int_0^{\pi} f(y)\,dy - \int_0^{\pi} y f(y)\,dy$$
$$= \pi \int_0^{\pi} f(y)\,dy - I$$

$$I = \pi \int_0^{\pi} f(x)\,dx - I$$
$$2I = \pi \int_0^{\pi} f(x)\,dx$$
$$I = \frac{\pi}{2} \int_0^{\pi} f(x)\,dx$$

(iii) $\sin x = \sin(\pi - x)$ for all x

$$\int_0^{\pi} x \sin x\,dx = \frac{\pi}{2} \int_0^{\pi} \sin x\,dx$$
$$= \frac{\pi}{2} \left[-\cos x\right]_0^{\pi}$$
$$= \frac{\pi}{2} \left(-(-1) - (-1)\right)$$
$$= \pi$$

$\sin^3 x = \sin^3(\pi - x)$

$$\int_0^{\pi} x \sin^3 x\,dx = \frac{\pi}{2} \int_0^{\pi} \sin x (1 - \cos^2 x)\,dx$$
$$= \frac{\pi}{2} \int_0^{\pi} \left(\sin x - \sin x \cos^2 x\right) dx$$
$$= \frac{\pi}{2} \left[-\cos x + \tfrac{1}{3}\cos^3 x\right]_0^{\pi}$$
$$= \frac{\pi}{2} \left(-(-1) + (-\tfrac{1}{3}) - (-1 + \tfrac{1}{3})\right)$$
$$= \tfrac{2}{3}\pi$$