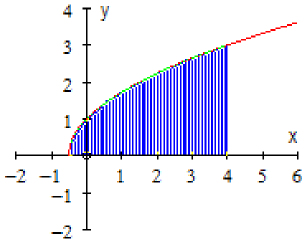
Section 2: Integration by substitution

Solutions to Exercise level 2

1.



$$\text{Area} = \int_{-\frac{1}{2}}^{4} \sqrt{1+2x}\,dx$$

Let $u = 1 + 2x \Rightarrow \dfrac{du}{dx} = 2 \Rightarrow dx = \frac{1}{2}du$

When $x = -\frac{1}{2}$, $u = 0$

When $x = 4$, $u = 9$

$$\begin{aligned}\text{Area} &= \int_{-\frac{1}{2}}^{4} \sqrt{1+2x}\,dx \\ &= \int_0^9 u^{\frac{1}{2}} \times \tfrac{1}{2}\,du \\ &= \left[\tfrac{1}{2} \times \tfrac{2}{3} u^{\frac{3}{2}}\right]_0^9 \\ &= \tfrac{1}{3}\left(9^{\frac{3}{2}} - 0\right) \\ &= \tfrac{1}{3} \times 27 \\ &= 9\end{aligned}$$

2. By inspection, $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx = \left(1+x^2\right)^{\frac{1}{2}} + c$

Let $u = \sqrt{1+x^2} \Rightarrow \dfrac{du}{dx} = \frac{1}{2}(1+x^2)^{-\frac{1}{2}} \times 2x \Rightarrow dx = \dfrac{\sqrt{1+x^2}}{x}du$

$$\begin{aligned}\int \frac{x}{\sqrt{1+x^2}}\,dx &= \int \frac{x}{\sqrt{1+x^2}} \times \frac{\sqrt{1+x^2}}{x}\,du \\ &= \int 1\,du \\ &= u + c \\ &= \sqrt{1+x^2} + c\end{aligned}$$

3. (i) $\displaystyle\int \frac{1}{2x+1}\,dx = \tfrac{1}{2}\ln|2x+1| + c$

(ii) Let $u = 1 + x^2 \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow dx = \dfrac{1}{2x}du$

$$\int \frac{3x}{1+x^2}\,dx = \int \frac{3x}{u} \times \frac{1}{2x}\,du$$
$$= \frac{3}{2}\int \frac{1}{u}\,du$$
$$= \tfrac{3}{2}\ln|u| + c$$
$$= \tfrac{3}{2}\ln(1+x^2) + c$$

4. (i) $\int \cot^2 2x\,dx = \int \left(\operatorname{cosec}^2 2x - 1\right)dx$

$$= -\tfrac{1}{2}\cot 2x - x + c$$

Using $1+\cot^2 x = \operatorname{cosec}^2 x$

(ii) $\int \frac{\tan x}{\sin 2x}\,dx = \int \frac{\frac{\sin x}{\cos x}}{2\sin x\cos x}\,dx$

$$= \int \frac{1}{2\cos^2 x}\,dx$$
$$= \int \tfrac{1}{2}\sec^2 x\,dx$$
$$= \tfrac{1}{2}\tan x + c$$

(iii) $\int \sin^2 3x\,dx = \int \tfrac{1}{2}(1-\cos 6x)\,dx$

Using $\cos 2x = 1 - 2\sin^2 x$

$$= \tfrac{1}{2}\left(x - \tfrac{1}{6}\sin 6x\right) + c$$
$$= \tfrac{1}{2}x - \tfrac{1}{12}\sin 6x + c$$

(iv) $\int (\sin 2x + \cos 2x)^2\,dx = \int \left(\sin^2 2x + 2\sin 2x\cos 2x + \cos^2 2x\right)dx$

$$= \int (1 + 2\sin 2x\cos 2x)\,dx$$
$$= \int (1 + \sin 4x)\,dx$$
$$= x - \tfrac{1}{4}\cos 4x + c$$

5. (i) $\int_0^{\frac{\pi}{4}} \sin^3 2x\,dx = \int_0^{\frac{\pi}{4}} \sin 2x(1-\cos^2 2x)\,dx$

$$= \int_0^{\frac{\pi}{4}} (\sin 2x - \sin 2x\cos^2 2x)\,dx$$
$$= \left[-\tfrac{1}{2}\cos 2x - \tfrac{1}{3}\cos^3 2x \times -\tfrac{1}{2}\right]_0^{\frac{\pi}{4}}$$
$$= \left[-\tfrac{1}{2}\cos 2x + \tfrac{1}{6}\cos^3 2x\right]_0^{\frac{\pi}{4}}$$
$$= -\tfrac{1}{2}\cos\tfrac{\pi}{2} + \tfrac{1}{6}\cos^3\tfrac{\pi}{2} - \left(-\tfrac{1}{2}\cos 0 + \tfrac{1}{6}\cos^3 0\right)$$
$$= 0 + \tfrac{1}{2} - \tfrac{1}{6}$$
$$= \tfrac{1}{3}$$

(ii) $\int_0^{\frac{\pi}{12}} \cos^4 x\,dx = \int_0^{\frac{\pi}{12}} (\tfrac{1}{2}(\cos 2x + 1))^2 dx$

$= \tfrac{1}{4}\int_0^{\frac{\pi}{12}} (\cos^2 2x + 2\cos 2x + 1)\,dx$

$= \tfrac{1}{4}\int_0^{\frac{\pi}{12}} (\tfrac{1}{2}(1 + \cos 4x) + 2\cos 2x + 1)\,dx$

$= \tfrac{1}{4}\int_0^{\frac{\pi}{12}} (\tfrac{1}{2}\cos 4x + 2\cos 2x + \tfrac{3}{2})\,dx$

$= \tfrac{1}{4}\left[\tfrac{1}{2} \times \tfrac{1}{4}\sin 4x + 2 \times \tfrac{1}{2}\sin 2x + \tfrac{3}{2}x\right]_0^{\frac{\pi}{12}}$

$= \tfrac{1}{4}(\tfrac{1}{8}\sin\tfrac{\pi}{3} + \sin\tfrac{\pi}{6} + \tfrac{3}{2} \times \tfrac{\pi}{12}) - 0$

$= \tfrac{1}{32} \times \tfrac{1}{2}\sqrt{3} + \tfrac{1}{4} \times \tfrac{1}{2} + \tfrac{1}{32}\pi$

$= \tfrac{1}{64}\sqrt{3} + \tfrac{1}{8} + \tfrac{1}{32}\pi$

(iii) $\int_0^{\frac{\pi}{3}} \tan\tfrac{1}{2}x\,dx = \int_0^{\frac{\pi}{3}} \frac{\sin\frac{1}{2}x}{\cos\frac{1}{2}x}\,dx = -2\int_0^{\frac{\pi}{3}} \frac{-\frac{1}{2}\sin\frac{1}{2}x}{\cos\frac{1}{2}x}\,dx$

$= -2\left[\ln\left|\cos\tfrac{1}{2}x\right|\right]_0^{\frac{\pi}{3}}$

$= -2\ln\left|\cos\tfrac{\pi}{6}\right| + 2\ln\left|\cos 0\right|$

$= -2\ln\left(\tfrac{1}{2}\sqrt{3}\right) + 2\ln 1$

$= -2\ln\left(\tfrac{1}{2}\sqrt{3}\right)$

(iv) $\int_0^{\pi/2} (1 + \cos x)^3 dx = \int_0^{\pi/2} (1 + 3\cos x + 3\cos^2 x + \cos^3 x)\,dx$

$= \int_0^{\pi/2} (1 + 3\cos x + 3 \times \tfrac{1}{2}(\cos 2x + 1) + \cos x(1 - \sin^2 x))\,dx$

$= \int_0^{\pi/2} (\tfrac{5}{2} + 4\cos x + \tfrac{3}{2}\cos 2x - \cos x \sin^2 x)\,dx$

$= \left[\tfrac{5}{2}x + 4\sin x + \tfrac{3}{4}\sin 2x - \tfrac{1}{3}\sin^3 x\right]_0^{\pi/2}$

$= \tfrac{5\pi}{4} + 4 + 0 - \tfrac{1}{3} - 0$

$= \tfrac{5\pi}{4} + \tfrac{11}{3}$

6. $\int_0^1 \frac{x^2}{x^3 + 2}\,dx = \left[\tfrac{1}{3}\ln\left|x^3 + 2\right|\right]_0^1$

$= \tfrac{1}{3}\ln 3 - \tfrac{1}{3}\ln 2$

$= \tfrac{1}{3}\ln\tfrac{3}{2}$

7. (i) By inspection $\int \frac{1 - 2x}{1 + x - x^2}\,dx = \ln(1 + x - x^2) + c$

(ii) By inspection $\int \left(1 + e^x\right)e^x dx = \tfrac{1}{2}\left(1 + e^x\right)^2 + c$

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8. $\frac{d}{dx}(x\ln x) = x \times \frac{1}{x} + \ln x = 1 + \ln x$

From above, $\int (1+\ln x)dx = x\ln x + c$

$$x + \int \ln x\,dx = x\ln x + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

9. (i) $\int \frac{\cos x}{1+2\sin x}\,dx = \tfrac{1}{2}\ln|1+2\sin x| + c$

(ii) $\int_0^1 \frac{1+e^x}{x+e^x}dx = \left[\ln\left|x+e^x\right|\right]_0^1$

$$= \ln(1+e) - \ln 1$$

$$= \ln(1+e)$$

10. $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{1}{\sin x}du$

$$\int \cos^2 x \sin x\;dx = \int u^2 \sin x \times -\frac{1}{\sin x}du$$

$$= -\int u^2 du$$

$$= -\tfrac{1}{3}u^3 + c$$

$$= -\tfrac{1}{3}\cos^3 x + c$$

11. Area $= \int_0^1 2^x dx$

$$= \int_0^1 e^{x\ln 2}dx$$

$$= \left[\frac{1}{\ln 2}e^{x\ln 2}\right]_0^1$$

$$= \frac{1}{\ln 2}\left(e^{\ln 2} - 1\right)$$

$$= \frac{1}{\ln 2}(2-1)$$

$$= \frac{1}{\ln 2}$$