Section 1: Solving equations numerically

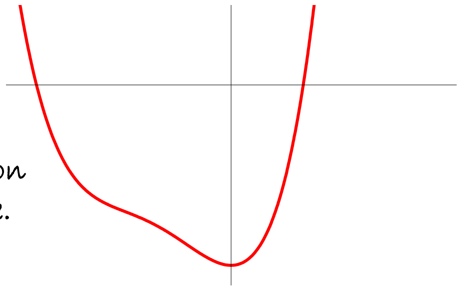
Solutions to Exercise level 2

1. (i) $f(x) = x^4 + 4x^3 + 5x^2 - 9$

$f'(x) = 4x^3 + 12x^2 + 10x = 2x(2x^2 + 6x + 5)$

The discriminant of $2x^2 + 6x + 5$ is $6^2 - 4\times 2\times 5 = 36 - 40 < 0$, so $2x^2 + 6x + 5$ is never zero.

Therefore the only turning point of the graph $y = x^4 + 4x^3 + 5x^2 - 9$ is when $x = 0$ and has coordinates (0, -9). Since f(x) is positive for large positive and negative values of x, the equation must have two roots, one positive and one negative.



(ii) $x^4 + 4x^3 + 5x^2 - 9 = 0$

$x^4 + 4x^3 + 5x^2 = 9$

$x^2(x^2 + 4x + 5) = 9$

$x^2 = \frac{9}{x^2 + 4x + 5}$

$x = \sqrt{\frac{9}{x^2 + 4x + 5}}$

(iii) $f(0) = -9$

$f(1) = 1 + 4 + 5 - 9 = 1$

so there is a root near $x = 1$.

$x_{n+1} = \sqrt{\frac{9}{x_n^2 + 4x_n + 5}}$

$x_0 = 1$

$x_1 = \sqrt{\frac{9}{x_0^2 + 4x_0 + 5}} = \sqrt{\frac{9}{1 + 4 + 5}} = 0.94868$

$x_2 = \sqrt{\frac{9}{x_1^2 + 4x_1 + 5}} = 0.96350$

$x_3 = \sqrt{\frac{9}{x_2^2 + 4x_2 + 5}} = 0.95918$

The root is 0.96 to 2 d.p.

2. (i) $f(x) = x^3 - x - 4$

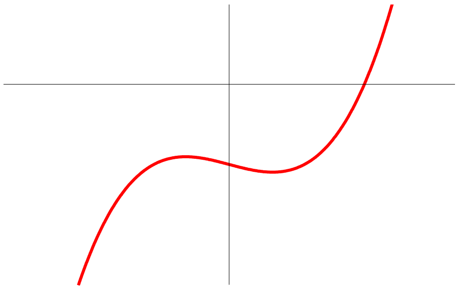
$f'(x) = 3x^2 - 1$

At turning points, $3x^2 - 1 = 0 \quad \Rightarrow x = \pm\frac{1}{\sqrt{3}}$

When $x = \frac{1}{\sqrt{3}}$, $f(x) = \frac{1}{3\sqrt{3}} - \frac{1}{\sqrt{3}} - 4 < 0$

When $x = -\frac{1}{\sqrt{3}}$, $f(x) = -\frac{1}{3\sqrt{3}} + \frac{1}{\sqrt{3}} - 4 < 0$

Both turning points are below the x-axis, so the graph cuts the x-axis only once.



Therefore the equation has only one root.

$f(1) = 1 - 1 - 4 = -4$

$f(2) = 8 - 2 - 4 = 2$

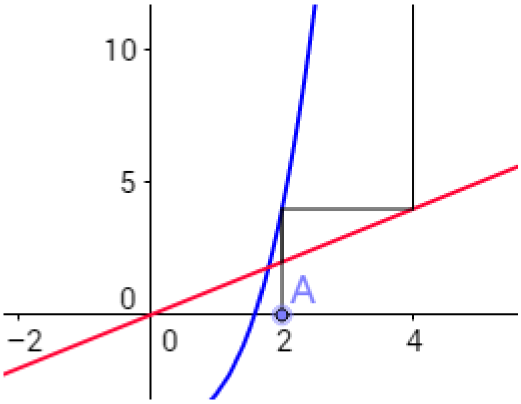
so the root lies between 1 and 2.

(ii) (a) $x_{n+1} = x_n^3 - 4$

$x_0 = 2$

$x_1 = 2^3 - 4 = 4$

$x_2 = 4^3 - 4 = 60$ - diverges

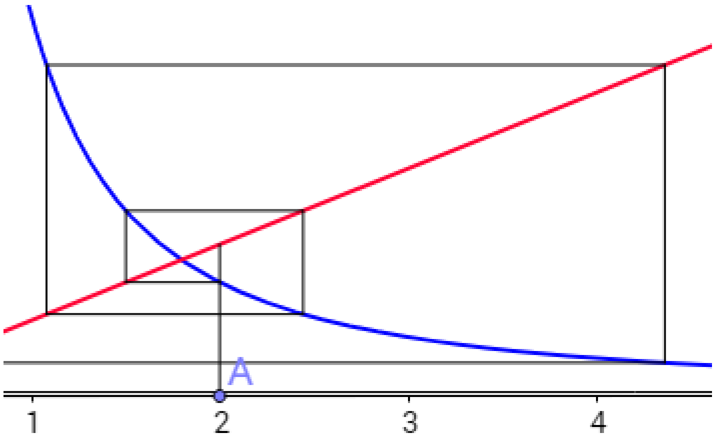


(b) $x_{n+1} = \frac{x_n + 4}{x_n^2}$

$x_0 = 2$

$x_1 = \frac{2+4}{2^2} = 1.5$

$x_2 = \frac{1.5+4}{1.5^2} = 2.444$ - diverges

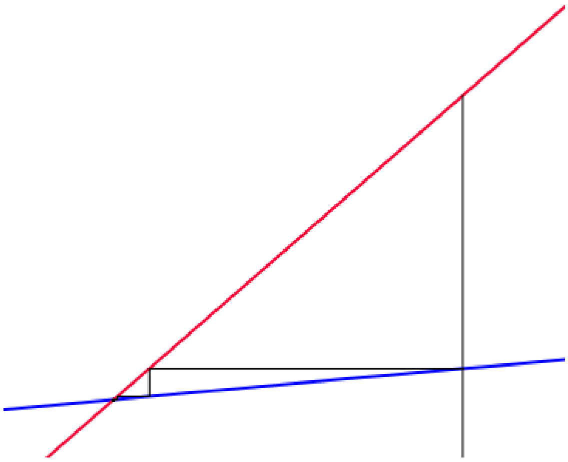


(c) $x_{n+1} = (x_n + 4)^{1/3}$

$x_0 = 2$

$x_1 = (2+4)^{1/3} = 1.81712$

$x_2 = (1.81712+4)^{1/3} = 1.79847$ - converges



7. (i) (a) $f(1) = -3$

$f(2) = 2$

so the root lies between 1 and 2

(b) $f(x) = x^3 - 2x - 2 \quad \Rightarrow f'(x) = 3x^2 - 2$

$$x_{n+1} = x_n - \frac{x^3 - 2x - 2}{3x^2 - 2}$$

Taking $x_0 = 2$: $x_1 = 1.8$

$x_2 = 1.769948187$

$x_3 = 1.769292663$

$x_4 = 1.769292354$

The root is 1.769 to 3 d.p.

Check: $f(1.7685) < 0$

$f(1.7695) > 0$

(ii) (a) $f(-2) = -3$

$f(-1) = 3$

The root lies between -2 and -1

(b) $f(x) = x^3 - x + 3 \quad \Rightarrow f'(x) = 3x^2 - 1$

$$x_{n+1} = x_n - \frac{x^3 - x + 3}{3x^2 - 1}$$

Taking $x_0 = -2 : x_1 = -1.727272727$

$x_2 = -1.673691174$

$x_3 = -1.67170257$

$x_4 = -1.671699882$

$x_5 = -1.671699882$

The root is -1.672 to 3 d.p.

Check: $f(-1.6715) > 0$

$f(-1.6725) < 0$

(iii) (a) $f(-1) = -2$

$f(0) = 1$

The root lies between -1 and 0

(b) $f(x) = 2x^5 - x^2 + 1 \quad f'(x) = 10x^4 - 2x$

$$x_{n+1} = x_n - \frac{2x^5 - x^2 + 1}{10x^4 - 2x}$$

Taking $x_0 = -1 : x_1 = -0.8333333333$

$x_2 = -0.7565596512$

$x_3 = -0.7423377915$

$x_4 = -0.7419128372$

$x_5 = -0.7419124700$

The root is -0.742 to 3 d.p.

Check: $f(-0.7415) > 0$

$f(-0.7425) < 0$

(iv) (a) $f(1) = -1$

$f(2) = 29$

The root lies between 1 and 2

(b) $f(x) = x^5 - x - 1 \quad \Rightarrow f'(x) = 5x^4 - 1$

$$x_{n+1} = x_n - \frac{x^5 - x - 1}{5x^4 - 1}$$

Taking $x_0 = 1$: $x_1 = 1.25$

$x_2 = 1.178459394$

$x_3 = 1.167537389$

$x_4 = 1.167304083$

$x_5 = 1.167303978$

The root is 1.167 to 3 d.p.

Check: $f(1.1665) < 0$

$f(1.1675) > 0$

(v) (a) $f(0) = -3$

$f(1) = 2$

The root lies between 0 and 1

(b) $f(x) = 3x^3 + 2x^2 - 3 \quad f'(x) = 9x^2 + 4x$

$$x_{n+1} = x_n - \frac{3x^3 + 2x^2 - 3}{9x^2 + 4x}$$

Taking $x_0 = 1$: $x_1 = 0.8461538462$

$x_2 = 0.8207752512$

$x_3 = 0.8201178710$

$x_4 = 0.8201174365$

The root is 0.820 to 3 d.p.

Check: $f(0.8195) < 0$

$f(0.8205) > 0$