Section 1: The shape of curves

Solutions to Exercise level 3

1. (i) $y = ax^3 + bx^2 + cx + d$

$\dfrac{dy}{dx} = 3ax^2 + 2bx + c$

$\dfrac{d^2y}{dx^2} = 6ax + 2b$

$\dfrac{d^2y}{dx^2} = 0 \quad \Rightarrow x = -\dfrac{b}{3a}$

When $x < -\dfrac{b}{3a}$, $\dfrac{d^2y}{dx^2} < 0$, and when $x > -\dfrac{b}{3a}$, $\dfrac{d^2y}{dx^2} > 0$. So there is a change of sign in $\dfrac{d^2y}{dx^2}$, and therefore the point where $x = -\dfrac{b}{3a}$ is a point of inflection.

When $x = -\dfrac{b}{3a}$, $y = a\left(-\dfrac{b}{3a}\right)^3 + b\left(-\dfrac{b}{3a}\right)^2 + c\left(-\dfrac{b}{3a}\right) + d$

$$= -\frac{b^3}{27a^2} + \frac{b^3}{9a^2} - \frac{bc}{3a} + d$$

$$= \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

So the point of inflection is $\left(-\dfrac{b}{3a}, \dfrac{2b^3}{27a^2} - \dfrac{bc}{3a} + d\right)$

(ii) If it is a stationary point, $\dfrac{dy}{dx} = 0 \Leftrightarrow 3ax^2 + 2bx + c = 0$

Putting $x = -\dfrac{b}{3a}$, $3a\left(-\dfrac{b}{3a}\right)^2 + 2b\left(-\dfrac{b}{3a}\right) + c = 0$

$$\Leftrightarrow \frac{b^2}{3a} - \frac{2b^2}{3a} + c = 0$$

$$\Leftrightarrow \frac{b^2}{3a} = c$$

$$\Leftrightarrow b^2 = 3ac$$

Edexcel A level Maths Differentiation 1 Exercise solns

2. $y = ax^4 + bx^3 + cx^2 + dx + e$

$$\frac{dy}{dx} = 4ax^3 + 3bx^2 + 2cx + d$$

$$\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 2c$$

If $\frac{d^2y}{dx^2} = 0$, $12ax^2 + 6bx + 2c = 0$

$$6ax^2 + 3bx + c = 0$$

This equation has no real roots if $9b^2 - 4 \times 6a \times c < 0$

$$\Rightarrow 3b^2 < 8ac$$

so if $3b^2 < 8ac$ there are no points of inflection.

If $3b^2 = 8ac$, the equation $6ax^2 + 3bx + c = 0$ has a repeated root, and therefore the sign of $\frac{d^2y}{dx^2}$ does not change. So in this case the point at which $\frac{d^2y}{dx^2} = 0$ is not a point of inflection.

So if $3b^2 \le 8ac$ there are no points of inflection.

3. There are points of inflection at $x = 1$ and $x = -2$, so the second derivative is zero at these points. The second derivative of a quartic is a quadratic, so

$$\frac{d^2y}{dx^2} = a(x-1)(x+2)$$

$$= a(x^2 + x - 2)$$

Integrating: $\frac{dy}{dx} = a(\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + c)$

The first derivative is zero at $x = 1$

so $a(\frac{1}{3} + \frac{1}{2} - 2 + c) = 0$

$a(c - \frac{7}{6}) = 0$

$c = \frac{7}{6}$

Integrating: $y = a(\frac{1}{12}x^4 + \frac{1}{6}x^3 - x^2 + \frac{7}{6}x + k)$

Curve passes through (1, 37): $a(\frac{1}{12} + \frac{1}{6} - 1 + \frac{7}{6} + k) = 37$

$ak + \frac{5}{12}a = 37 \quad (1)$

Curve passes through (-2, -125): $a(\frac{4}{3} - \frac{4}{3} - 4 - \frac{7}{3} + k) = -125$

$ak - \frac{19}{3}a = -125 \quad (2)$

(1) – (2) $\Rightarrow \frac{81}{12}a = 162$

$\Rightarrow a = 24$

$ak + \frac{5}{12}a = 37 \Rightarrow 24k + 10 = 37 \Rightarrow k = \frac{27}{24} = \frac{9}{8}$

The equation of the curve is $y = 24(\frac{1}{12}x^4 + \frac{1}{6}x^3 - x^2 + \frac{7}{6}x + \frac{9}{8})$

$$= 2x^4 + 4x^3 - 24x^2 + 28x + 27$$