Section 1: The shape of curves

Solutions to Exercise level 1

1. (i) $y = x^3 + 2x^2 - 1$

$\frac{dy}{dx} = 3x^2 + 4x$

$\frac{d^2y}{dx^2} = 6x + 4$

When $x = 1$, $\frac{d^2y}{dx^2} > 0$ so the curve is convex at this point.

(ii) $y = 2x^3 - 3x^2 + 4x$

$\frac{dy}{dx} = 6x^2 - 6x + 4$

$\frac{d^2y}{dx^2} = 12x - 6$

When $x = 0$, $\frac{d^2y}{dx^2} < 0$ so the curve is concave at this point.

(iii) $y = x^4 + 3x^2 - 1$

$\frac{dy}{dx} = 4x^3 + 6x$

$\frac{d^2y}{dx^2} = 12x^2 + 6$

When $x = -2$, $\frac{d^2y}{dx^2} > 0$ so the curve is convex at this point.

2. (i) $y = x^3 - 3x^2 + 2x + 1$

$\frac{dy}{dx} = 3x^2 - 6x + 2$

$\frac{d^2y}{dx^2} = 6x - 6$

At point of inflection, $\frac{d^2y}{dx^2} = 0$

$6x - 6 = 0$

$x = 1$

When $x = 1, y = 1 - 3 + 2 + 1 = 1$

so the point of inflection is $(1, 1)$

(ii) $y = 3x^3 - 4x + 2$

$$\frac{dy}{dx} = 9x^2 - 4$$

$$\frac{d^2y}{dx^2} = 18x$$

At point of inflection, $\frac{d^2y}{dx^2} = 0$

$$18x = 0$$

$$x = 0$$

When $x = 0, y = 2$

so the point of inflection is $(0, 2)$

(iii) $y = x^4 + 3x^3 - 6x^2 + 2x - 1$

$$\frac{dy}{dx} = 4x^3 + 9x^2 - 12x + 2$$

$$\frac{d^2y}{dx^2} = 12x^2 + 18x - 12$$

At point of inflection, $\frac{d^2y}{dx^2} = 0$

$$12x^2 + 18x - 12 = 0$$

$$2x^2 + 3x - 2 = 0$$

$$(2x - 1)(x + 2) = 0$$

$$x = \tfrac{1}{2} \text{ or } -2$$

When $x = \frac{1}{2}, y = \frac{1}{16} + \frac{3}{8} - \frac{3}{2} + 1 - 1 = -\frac{17}{16}$

When $x = -2, y = 16 - 24 - 24 - 4 - 1 = -37$

The points of inflection are $\left(\frac{1}{2}, -\frac{17}{16}\right)$ and (-2, -37).

3. $y = x^3 + 3x^2 + 3x + 4$

$$\frac{dy}{dx} = 3x^2 + 6x + 3$$

At stationary points, $\frac{dy}{dx} = 0$

$$3x^2 + 6x + 3 = 0$$

$$x^2 + 2x + 1 = 0$$

$$(x + 1)^2 = 0$$

$$x = -1$$

When $x < -1, \frac{dy}{dx} > 0$

When $x > -1, \frac{dy}{dx} > 0$

so the stationary point is a point of inflection.

When $x = -1, y = -1 + 3 - 3 + 4 = 3$

so the stationary point of inflection is (-1,3)