Section 2: Parametric differentiation

Solutions to Exercise level 3

1. At (6, 8) $9-t^2=8 \quad \Rightarrow t=\pm 1$

$t=1 \quad \Rightarrow x=8-2=6$

$t=-1 \quad \Rightarrow x=-8+2=-6$

so (6, 8) is the point where $t=1$

$x=8t-2t^3 \quad \Rightarrow \frac{dx}{dt}=8-6t^2$

$y=9-t^2 \quad \Rightarrow \frac{dy}{dt}=-2t$

$\frac{dy}{dx}=\frac{-2t}{8-6t^2}=\frac{t}{3t^2-4}$

At $t=1$, gradient $=\frac{1}{3-4}=-1$

so gradient of normal = 1

Equation of normal is $y-8=1(x-6)$

$y=x+2$

At points where normal intersects curve, $9-t^2=8t-2t^3+2$

$2t^3-t^2-8t+7=0$

Normal intersects curve at $t=1$, so $t-1$ is a factor

$(t-1)(2t^2+t-7)=0$

$2t^2+t-7=0$ has discriminant $1^2-4\times2\times7=57$, so as this is positive the quadratic has two distinct real roots and so the normal intersects C at exactly two other points.

$2t^2+t-7=0 \quad \Rightarrow t=\frac{-1\pm\sqrt{57}}{4}$

2. Where curve crosses x-axis, $\sin t+\sin 2t=0$

$\sin t+2\sin t\cos t=0$

$\sin t(1+2\cos t)=0$

$\sin t=0 \Rightarrow t=0, \pi$

$\cos t=-\tfrac{1}{2} \Rightarrow t=\frac{2\pi}{3}, \frac{4\pi}{3}$

When $t=0$, $\quad x=\cos 0+\cos 0=1+1=2$

$y=\sin 0+\sin 0=0+0=0$

When $t=\pi$, $\quad x=\cos\pi+\cos 2\pi=-1+1=0$

$y=\sin\pi+\sin 2\pi=0+0=0$

When $t=\frac{2\pi}{3}$, $\quad x=\cos\frac{2\pi}{3}+\cos\frac{4\pi}{3}=-\frac{1}{2}-\frac{1}{2}=-1$

$$y = \sin\frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = 0$$

When $t = \frac{4\pi}{3}$, $\quad x = \cos\frac{4\pi}{3} + \cos\frac{8\pi}{3} = -\frac{1}{2} - \frac{1}{2} = -1$

$$y = \sin\frac{4\pi}{3} + \sin\frac{8\pi}{3} = -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = 0$$

The points are (2, 0), (0, 0) and (-1, 0). The curve crosses itself at (-1, 0) as there are two values of t that give this point.

$$x = \cos t + \cos 2t \quad \Rightarrow \frac{dx}{dt} = -\sin t - 2\sin 2t$$

$$y = \sin t + \sin 2t \quad \Rightarrow \frac{dy}{dt} = \cos t + 2\cos 2t$$

$$\frac{dy}{dx} = \frac{\cos t + 2\cos 2t}{-\sin t - 2\sin 2t}$$

When $t = \frac{2\pi}{3}$, $\frac{dy}{dx} = \frac{\cos\frac{2\pi}{3} + 2\cos\frac{4\pi}{3}}{-\sin\frac{2\pi}{3} - 2\sin\frac{4\pi}{3}} = \frac{-\frac{1}{2} + 2\times -\frac{1}{2}}{-\frac{1}{2}\sqrt{3} + \sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$

Tangent is $y = -\sqrt{3}(x+1)$

When $t = \frac{4\pi}{3}$, $\frac{dy}{dx} = \frac{\cos\frac{4\pi}{3} + 2\cos\frac{8\pi}{3}}{-\sin\frac{4\pi}{3} - 2\sin\frac{8\pi}{3}} = \frac{-\frac{1}{2} + 2\times -\frac{1}{2}}{\frac{1}{2}\sqrt{3} - \sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

Tangent is $y = \sqrt{3}(x+1)$

3. (i) When $y = 0$ $\qquad t^3 - 3t = 0$

$$t(t^2 - 3) = 0$$

$$t = 0, \pm\sqrt{3}$$

When $t = 0, x = 0$ so this is not the point (3, 0)

When $t = \pm\sqrt{3}, x = 3$ so these two values of t both give (3, 0)

$$x = t^2 \quad \Rightarrow \frac{dx}{dt} = 2t$$

$$y = t^3 - 3t \quad \Rightarrow \frac{dy}{dt} = 3t^2 - 3$$

$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$$

When $t = \sqrt{3}$, gradient $= \frac{3\times 3 - 3}{2\sqrt{3}} = \frac{6}{2\sqrt{3}} = \sqrt{3}$

Tangent is $y = \sqrt{3}(x - 3)$

When $t = -\sqrt{3}$, gradient $= \frac{3\times 3 - 3}{-2\sqrt{3}} = -\frac{6}{2\sqrt{3}} = -\sqrt{3}$

Tangent is $y = -\sqrt{3}(x-3)$

(ii) When the tangent is horizontal, $\frac{dy}{dt} = 0$

$$3t^2 - 3 = 0$$
$$t = \pm 1$$

When $t = 1$, $x = 1$ and $y = 1 - 3 = -2$

When $t = -1$, $x = 1$ and $y = -1 + 3 = 2$

so the tangent is horizontal at (1, -2) and (1, 2)

When the tangent is vertical, $\frac{dx}{dt} = 0$

$$2t = 0$$
$$t = 0$$

When $t = 0$, $x = 0$ and $y = 0$

so the tangent is vertical at (0, 0)

The tangent at (1, -2) has equation $y = -2$

Where this meets the curve, $t^3 - 3t = -2$

$$t^3 - 3t + 2 = 0$$
$$(t-1)(t^2 + t - 2) = 0$$
$$(t-1)(t-1)(t+2) = 0$$
$$t = 1 \text{ or } -2$$

The tangent meets the curve at (1, -2) and (4, -2)

The tangent at (1, 2) has equation $y = 2$

Where this meets the curve, $t^3 - 3t = 2$

$$t^3 - 3t - 2 = 0$$
$$(t+1)(t^2 - t - 2) = 0$$
$$(t+1)(t+1)(t-2) = 0$$
$$t = -1 \text{ or } 2$$

The tangent meets the curve at (1, 2) and (4, 2)

The tangent at (0, 0) has equation $x = 0$

Where this meets curve, $t^2 = 0 \quad \Rightarrow t = 0$

so this meets the curve at (0, 0) only

(iii) If $t = a$, $x = a^2$ and $y = a^3 - 3a$

If $t=-a$, $x=a^2$ and $y=(-a)^3-3(-a)=-(a^3-3a)$

So every point (a^2, a^3-3a) has a mirror image in the x-axis,

$(a^2, -(a^3-3a))$ and so the curve is symmetric in the x-axis

(iv)