Section 2: Parametric differentiation and integration

Solutions to Exercise level 2

1. When curve crosses x-axis, $y = 0$ so $t^2 - 1 = 0$

$$(t+1)(t-1) = 0$$
$$t = -1 \text{ or } t = 1$$

$x = t(t^2+1) = t^3 + t \quad \Rightarrow \dfrac{dx}{dt} = 3t^2 + 1$

$y = t^2 - 1 \quad \Rightarrow \dfrac{dy}{dt} = 2t$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t}{3t^2+1}$

When $t = 1$, $\dfrac{dy}{dx} = \dfrac{-2}{3+1} = -\frac{1}{2}$

When $t = 1$, $x = 1(1^2+1) = 2$

So the tangent has gradient $\frac{1}{2}$ and passes through the point (2, 0)

Equation of tangent is $y = \frac{1}{2}(x-2)$

$$2y = x - 2$$

When $t =$ -1, $\dfrac{dy}{dx} = \dfrac{2}{3+1} = \frac{1}{2}$

When $t =$ -1, $x = -1((-1)^2+1) = -2$

So the tangent has gradient $-\frac{1}{2}$ and passes through the point (-2, 0)

Equation of tangent is $y = -\frac{1}{2}(x-(-2))$

$$2y = -x - 2$$
$$2y + x + 2 = 0$$

2. $x = t^2 \quad \Rightarrow \dfrac{dx}{dt} = 2t$

$y = 2t \quad \Rightarrow \dfrac{dy}{dt} = 2$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2}{2t} = \dfrac{1}{t}$

Tangent has gradient $\dfrac{1}{t}$ and passes through the point $(t^2, 2t)$.

Equation of tangent is $y - 2t = \dfrac{1}{t}(x - t^2)$

$$ty - 2t^2 = x - t^2$$
$$ty = x + t^2$$

At point A, $y=0 \quad \Rightarrow x+t^2=0 \quad \Rightarrow x=-t^2$

At point B, $x=0 \quad \Rightarrow ty=t^2 \quad \Rightarrow y=t$

The coordinates of A are $(-t^2,0)$ and the coordinates of B are $(0,t)$.

3. $x=4t \quad \Rightarrow \frac{dx}{dt}=4$

$y=\frac{4}{t}=4t^{-1} \quad \Rightarrow \frac{dy}{dt}=-4t^{-2}$

$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-4t^{-2}}{4}=-\frac{1}{t^2}$

At the point P (8, 2), t = 2

Gradient at P $=-\frac{1}{4}$.

Gradient of normal at P = 4

Normal has gradient 4 and passes through (8, 2)

Gradient of normal is $y-2=4(x-8)$

$$y-2=4x-32$$

$$y=4x-30$$

To find the point where the normal meets the curve again, substitute the parametric equations for x and y into the equation of the normal.

$$y=4x-30$$

$$\frac{4}{t}=4\times 4t-30$$

$$4=16t^2-30t$$

$$8t^2-15t-2=0$$

$$(8t+1)(t-2)=0$$

$$t=-\tfrac{1}{8} \text{ or } t=2$$

Since $t=2$ at point P, then at Q $t=-\frac{1}{8}$.

At Q, $x=4\times-\frac{1}{8}=-\frac{1}{2}$

$$y=\frac{4}{-\frac{1}{8}}=-32$$

The coordinates of Q are $(-\frac{1}{2},-32)$.

4. (i) At intersections with x-axis, $y=0 \quad \Rightarrow 3\sin\theta=0 \quad \Rightarrow \theta=0$ or π

When $\theta=0, \quad x=2\cos 0=2$

When $\theta=0, \quad x=2\cos\pi=-2$

At intersections with y-axis, $x=0 \quad \Rightarrow 2\cos\theta=0 \quad \Rightarrow \theta=\frac{\pi}{2}$ or $\frac{3\pi}{2}$

When $\theta=\frac{\pi}{2}, \quad y=3\sin\frac{\pi}{2}=3$

When $\theta = \frac{3\pi}{2}$, $\quad y = 3\sin\frac{3\pi}{2} = -3$

The points of intersection with the axes are (2, 0), (-2, 0), (0, 3), (0, -3)

(ii) $x = 2\cos\theta \quad \Rightarrow \dfrac{dx}{dt} = -2\sin\theta$

$y = 3\sin\theta \quad \Rightarrow \dfrac{dy}{dt} = 3\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta}{-2\sin\theta}$$

Equation of tangent is $y - 3\sin\theta = -\dfrac{3\cos\theta}{2\sin\theta}(x - 2\cos\theta)$

Tangent passes through the point (4, 0)

$$-\cancel{3}\sin\theta = -\frac{\cancel{3}\cos\theta}{2\sin\theta}(4 - 2\cos\theta)$$

$$2\sin^2\theta = \cos\theta(4 - 2\cos\theta)$$

$$2\sin^2\theta = 4\cos\theta - 2\cos^2\theta$$

$$\sin^2\theta + \cos^2\theta = 2\cos\theta$$

$$\cos\theta = \tfrac{1}{2}$$

$$\theta = \tfrac{\pi}{3} \quad \text{or} \quad \theta = \tfrac{5\pi}{3}$$

When $\theta = \frac{\pi}{3}$, $x = 2\cos\frac{\pi}{3} = 2\times\frac{1}{2} = 1$

$y = 3\sin\frac{\pi}{3} = 3\times\frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3}$

When $\theta = \frac{5\pi}{3}$, $x = 2\cos\frac{5\pi}{3} = 2\times\frac{1}{2} = 1$

$y = 3\sin\frac{5\pi}{3} = 3\times -\frac{1}{2}\sqrt{3} = -\frac{3}{2}\sqrt{3}$

The points are $\left(1, \frac{3}{2}\sqrt{3}\right)$ and $\left(1, -\frac{3}{2}\sqrt{3}\right)$.

5. (i) $x = 2(\theta - \sin\theta) \quad \Rightarrow \dfrac{dx}{d\theta} = 2 - 2\cos\theta$

$y = 2(1 - \cos\theta) \quad \Rightarrow \dfrac{dy}{d\theta} = 2\sin\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\sin\theta}{2 - 2\cos\theta} = \frac{\sin\theta}{1 - \cos\theta}$$

(ii) When $\theta = \frac{\pi}{2}$, gradient $= \dfrac{\sin\frac{\pi}{2}}{1 - \cos\frac{\pi}{2}} = \dfrac{1}{1 - 0} = 1$

Gradient of normal = -1

When $\theta = \frac{\pi}{2}$, $x = 2(\frac{\pi}{2} - 1) = \pi - 2$

$y = 2(1 - 0) = 2$

Equation of normal is $y-2=-1(x-(\pi-2))$

$y-2=-x+\pi-2$

$y=-x+\pi$

When normal meets x-axis, $y=0 \Rightarrow x=\pi$.

Coordinates of A are $(\pi,0)$.

(iii) $x=\pi$

Gradient of tangent at P is 1

Tangent at P is $y-2=1(x-(\pi-2))$

$y-2=x-\pi+2$

$y=x+4-\pi$

At point of intersection with $x=\pi$

$y=\pi+4-\pi=4$

so the point of intersection is $(\pi,4)$.

6. (i) When $y=0$, $2t(1-t^2)=0$

$2t(1-t)(1+t)=0$

$t=0$ or $t=1$ or $t=-1$

When $t=0$, $x=0$

When $t=\pm1$, $x=4$

so the area in the loop is twice the area from t = 0 to t = 1.

$x=4t^2 \Rightarrow \frac{dx}{dt}=8x$

Area $=2\int_0^1 y\frac{dx}{dt}dt=2\int_0^1 2t(1-t^2)\times 8t\,dt$

$=32\int_0^1 (t^2-t^4)\,dt$

$=32\left[\frac{1}{3}t^3-\frac{1}{5}t^5\right]_0^1$

$=32\left(\frac{1}{3}-\frac{1}{5}\right)$

$=32\times\frac{2}{15}=\frac{64}{15}$

(ii) $x=4t^2 \Rightarrow t=\frac{1}{2}\sqrt{x}$

$y=2t(1-t^2)$

$=2\times\frac{1}{2}\sqrt{x}\left(1-\frac{1}{4}x\right)$

$=\sqrt{x}\left(1-\frac{1}{4}x\right)$

(iii) $y=x^{\frac{1}{2}}-\frac{1}{4}x^{\frac{3}{2}}$

Area in loop is twice the area under the curve from x = 0 to x = 4.

$$\text{Area } = 2\int_0^4 \left(x^{\frac{1}{2}} - \tfrac{1}{4}x^{\frac{3}{2}}\right)dx$$
$$= 2\left[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{4}\times\tfrac{2}{5}x^{\frac{5}{2}}\right]_0^4$$
$$= 2\left(\tfrac{2}{3}\times 8 - \tfrac{1}{10}\times 32\right)$$
$$= 2\left(\tfrac{16}{3} - \tfrac{16}{5}\right)$$
$$= \tfrac{64}{15}$$

7. $x = \theta - \sin\theta, \quad y = 1 - \cos\theta$

The graph meets the x-axis when $y = 0 \quad \Rightarrow \cos\theta = 1 \quad \Rightarrow \theta = 0, 2\pi, 4\pi$ etc.

Area under one loop is from $\theta = 0$ to $\theta = 2\pi$.

$$\frac{dx}{d\theta} = 1 - \cos\theta$$

$$\text{Area } = \int_0^{2\pi} y\frac{dx}{d\theta}d\theta$$
$$= \int_0^{2\pi} (1-\cos\theta)(1-\cos\theta)d\theta$$
$$= \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)d\theta$$
$$= \int_0^{2\pi} (1 - 2\cos\theta + \tfrac{1}{2}(\cos 2\theta + 1))d\theta$$
$$= \int_0^{2\pi} (\tfrac{3}{2} - 2\cos\theta + \tfrac{1}{2}\cos 2\theta)d\theta$$
$$= \left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi}$$
$$= \tfrac{3}{2}\times 2\pi$$
$$= 3\pi$$