Section 2: Parametric differentiation and integration

Solutions to Exercise level 1

1. (i) $x = 3t - 2 \quad \Rightarrow \dfrac{dx}{dt} = 3$

$y = t^3 + 1 \quad \Rightarrow \dfrac{dy}{dt} = 3t^2$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2}{3} = t^2$

(ii) $x = 2\cos^3\theta \quad \Rightarrow \dfrac{dx}{d\theta} = 2 \times 3\cos^2\theta \times -\sin\theta = -6\sin\theta\cos^2\theta$

$y = 3\sin^3\theta \quad \Rightarrow \dfrac{dy}{d\theta} = 3 \times 3\sin^2\theta \times \cos\theta = 9\sin^2\theta\cos\theta$

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{9\sin^2\theta\cos\theta}{-6\sin\theta\cos^2\theta} = -\tfrac{3}{2}\tan\theta$

(iii) $x = \dfrac{1}{t^2} = t^{-2} \quad \Rightarrow \dfrac{dx}{dt} = -2t^{-3} = -\dfrac{2}{t^3}$

$y = 1 + t \quad \Rightarrow \dfrac{dy}{dt} = 1$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1}{-2/t^3} = -\tfrac{1}{2}t^3$

2. (i) When $t = 1$, $x = 1^3 - 2 \times 1 = -1$

$$y = 5 \times 1^2 + \frac{1}{1} = 6$$

so the coordinates of the point are (-1, 6)

(ii) $x = t^3 - 2t \quad \Rightarrow \dfrac{dx}{dt} = 3t^2 - 2$

$y = 5t^2 + \dfrac{1}{t} = 5t^2 + t^{-1} \quad \Rightarrow \dfrac{dy}{dt} = 10t - t^{-2}$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{10t - t^{-2}}{3t^2 - 2}$

(iii) When $t = 1$, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{10 - 1}{3 - 2} = 9$

(iv) Tangent has gradient 9 and passes through (-1, 6)

Equation of tangent is $y-6=9(x-(-1))$

$$y-6=9x+9$$
$$y=9x+15$$

Normal has gradient $-\frac{1}{9}$ and passes through (-1, 6)

Equation of normal is $y-6=-\frac{1}{9}(x-(-1))$

$$9(y-6)=-(x+1)$$
$$9y-54=-x-1$$
$$9y+x=53$$

3. (i) Where curve meets the x-axis, $t-t^2=0$

$$t(t-1)=0$$
$$t=0 \text{ or } t=1$$

(ii) $x=t+t^2 \quad \Rightarrow \frac{dx}{dt}=1+2t$

$$\text{Area } =\int_0^1 y\frac{dx}{dt}dt=\int_0^1 (t-t^2)(1+2t)dt$$
$$=\int_0^1 (t+t^2-2t^3)dt$$
$$=\left[\tfrac{1}{2}t^2+\tfrac{1}{3}t^3-\tfrac{1}{2}t^4\right]_0^1$$
$$=\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{2}-0$$
$$=\tfrac{1}{3} \text{ square units}$$