Section 1: Differentiating exponentials and logarithms

Exercise level 3

1. (i) Since $k = e^{\ln k}$ for any k:

$$2^x = e^{\ln(2^x)} = e^{x\ln 2}$$

(ii) $y = 2^x = e^{x\ln 2}$

$$\frac{dy}{dx} = e^{x\ln 2} \times \ln 2 = 2^x \ln 2$$

(iii) $y = x^x = e^{\ln(x^x)} = e^{x\ln x}$

$$\frac{dy}{dx} = \left(x \times \frac{1}{x} + 1 \times \ln x\right)e^{x\ln x}$$
$$= (1+\ln x)e^{x\ln x}$$
$$= (1+\ln x)x^x$$

At stationary point, $1 + \ln x = 1$

$$\ln x = -1$$
$$x = e^{-1} = \frac{1}{e}$$

When $x = \frac{1}{e}$, $y = \left(\frac{1}{e}\right)^{\frac{1}{e}}$, so the stationary point is $\left(\frac{1}{e}, \left(\frac{1}{e}\right)^{\frac{1}{e}}\right)$

2. (i) $y = \ln x$

$$x = e^y$$
$$\frac{dx}{dy} = e^y = x$$
$$\frac{dy}{dx} = \frac{1}{x}$$

(ii) $y = \ln(\ln x)$

Let $u = \ln x \quad \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$y = \ln u \quad \Rightarrow \frac{dy}{dx} = \frac{1}{u} = \frac{1}{\ln x}$

$$\frac{dy}{dx} = \frac{1}{\ln x} \times \frac{1}{x} = \frac{1}{x\ln x}$$

(iii) $y = \ln(\ln x^x) = \ln(x \ln x) = \ln x + \ln(\ln x)$

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x\ln x}$$
$$= \frac{\ln x + 1}{x \ln x}$$
$$= \frac{\ln x + \ln e}{x \ln x}$$
$$= \frac{\ln(ex)}{x \ln x}$$

3. (i) $f(x) = \ln(1+x) - x + \frac{1}{2}x^2$

$$f'(x) = \frac{1}{1+x} - 1 + x$$

(ii) $f'(x) = \frac{1}{1+x} - 1 + x$

$$= \frac{1 - (1+x) + x(1+x)}{1+x}$$
$$= \frac{1 - 1 - x + x + x^2}{1+x}$$
$$= \frac{x^2}{1+x}$$

Since $x > 0, 1 + x > 0$, and $x^2 > 0$, so $f'(x) > 0$ for $x > 0$

Since the gradient is positive for $x > 0$, f(x) is an increasing function

Since $f(0) = 0$, $f(x) > 0$ for $x > 0$

Therefore for $x > 0$, $\ln(1+x) - x + \frac{1}{2}x^2 > 0 \neq$

$\ln(1+x) > x - \frac{1}{2}x^2$

(iii) $g(x) = \ln(1+x) - x + \frac{1}{2}x^2 - \frac{1}{3}x^3$

$$g'(x) = \frac{1}{1+x} - 1 + x - x^2$$
$$= \frac{1 - (1+x) + x(1+x) - x^2(1+x)}{1+x}$$
$$= \frac{1 - 1 - x + x + x^2 - x^2 - x^3}{1+x} = -\frac{x^3}{1+x}$$

For $x > 0, 1 + x > 0$ and $x^3 > 0$, so $g'(x) < 0$ for $x > 0$

Since the gradient is negative for $x > 0$, g(x) is a decreasing function.

Since $g(0) = 0$, $g(x) < 0$ for $x > 0$

Therefore for $x > 0$, $\ln(1+x) - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 < 0$

$$\ln(1+x) < x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

(iv) Putting $x = \frac{1}{2}$:

$$\ln(1+\tfrac{1}{2}) > \tfrac{1}{2} - \tfrac{1}{2}\times(\tfrac{1}{2})^2$$

$$\ln\tfrac{3}{2} > \tfrac{1}{2} - \tfrac{1}{8}$$

$$\ln\tfrac{3}{2} > \tfrac{3}{8}$$

and

$$\ln(1+\tfrac{1}{2}) < \tfrac{1}{2} - \tfrac{1}{2}\times(\tfrac{1}{2})^2 + \tfrac{1}{3}\times(\tfrac{1}{2})^3$$

$$\ln(\tfrac{3}{2}) < \tfrac{1}{2} - \tfrac{1}{8} + \tfrac{1}{24}$$

$$\ln(\tfrac{3}{2}) < \tfrac{5}{12}$$

So $\dfrac{3}{8} < \ln\dfrac{3}{2} < \dfrac{5}{12}$.