Section 1: Differentiating exponentials and logarithms

Solutions to Exercise level 2

1. $f(x) = (1 + x^2)e^{3x}$

Let $u = 1 + x^2 \quad \Rightarrow \dfrac{du}{dx} = 2x$

Let $v = e^{3x} \quad \Rightarrow \dfrac{dv}{dx} = 3e^{3x}$

Using the product rule, $f'(x) = (1 + x^2) \times 3e^{3x} + 2xe^{3x}$

$= (3 + 2x + 3x^2)e^{3x}$

2. $y = e^{x^2} \ln x$

$u = e^{x^2} \quad \Rightarrow \dfrac{du}{dx} = 2xe^{x^2}$

$v = \ln x \quad \Rightarrow \dfrac{dv}{dx} = \dfrac{1}{x}$

Using the product rule, $\dfrac{dy}{dx} = \dfrac{1}{x}e^{x^2} + 2xe^{x^2} \ln x$

3. (i) $y = 2^x$

$\ln y = x \ln 2$

$x = \dfrac{\ln y}{\ln 2}$

$\dfrac{dx}{dy} = \dfrac{1}{y \ln 2}$

$\dfrac{dy}{dx} = y \ln 2 = 2^x \ln 2$

(ii) $y = 2^{3x} \quad \Rightarrow \dfrac{dy}{dx} = 3 \times 2^{3x} \ln 2$

4. $y = x \ln x$

$u = x \quad \Rightarrow \dfrac{du}{dx} = 1$

$v = \ln x \quad \Rightarrow \dfrac{dv}{dx} = \dfrac{1}{x}$

Using the product rule, $\dfrac{dy}{dx} = x \times \dfrac{1}{x} + 1 \times \ln x = 1 + \ln x$

At turning point, $\frac{dy}{dx} = 0$

$$1 + \ln x = 0$$

$$\ln x = -1$$

$$x = e^{-1} = \frac{1}{e}$$

When $x = \frac{1}{e}$, $y = \frac{1}{e}\ln e^{-1} = -\frac{1}{e}$

Turning point is $\left(\frac{1}{e}, -\frac{1}{e}\right)$

$\frac{dy}{dx} = 1 + \ln x \quad \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x}$

When $x = \frac{1}{e}$, $\frac{d^2y}{dx^2} = e > 0$, so the turning point is a local minimum.

5. $y = xe^{-2x}$

$u = x \quad \Rightarrow \frac{du}{dx} = 1$

$v = e^{-2x} \quad \Rightarrow \frac{dv}{dx} = -2e^{-2x}$

$\frac{dy}{dx} = x \times -2e^{-2x} + 1 \times e^{-2x} = (1-2x)e^{-2x}$

At turning point, $\frac{dy}{dx} = 0$

$$(1-2x)e^{-2x} = 0$$

$$x = \tfrac{1}{2}$$

When $x = \frac{1}{2}$, $y = \frac{1}{2}e^{-1} = \frac{1}{2e}$

Turning point is $\left(\frac{1}{2}, \frac{1}{2e}\right)$

$\frac{dy}{dx} = (1-2x)e^{-2x}$

$u = 1 - 2x \quad \Rightarrow \frac{du}{dx} = -2$

$v = e^{-2x} \quad \Rightarrow \frac{dv}{dx} = -2e^{-2x}$

Using the product rule, $\frac{d^2y}{dx^2} = (1-2x) \times -2e^{-2x} - 2e^{-2x} = (-4+4x)e^{-2x}$

When $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = -2e^{-2x} < 0$, so turning point is a local maximum.

6. $y = \frac{e^x}{\sqrt{1+2x}}$

$u = e^x \quad \Rightarrow \frac{du}{dx} = e^x$

$v = (1+2x)^{\frac{1}{2}} \Rightarrow \frac{dv}{dx} = \frac{1}{2}(1+2x)^{-\frac{1}{2}} \times 2 = (1+2x)^{-\frac{1}{2}}$

Using the quotient rule, $\frac{dy}{dx} = \frac{e^x(1+2x)^{\frac{1}{2}} - e^x(1+2x)^{-\frac{1}{2}}}{1+2x}$

$= \frac{e^x(1+2x) - e^x}{(1+2x)(1+2x)^{\frac{1}{2}}}$

$= \frac{2xe^x}{\sqrt{(1+2x)^3}}$

7. $T = 20 + 80e^{-0.5t}$

$\frac{dT}{dt} = 80 \times -0.5e^{-0.5t} = -40e^{-0.5t}$

When $t = 0$, $\frac{dT}{dt} = -40e^0 = -40$

The initial rate of cooling is 40 degrees / minute.

When $t = 2$, $\frac{dT}{dt} = -40e^{-0.5\times2} = -\frac{40}{e} = -14.7$

The rate of cooling after 2 minutes is 14.7 degrees / minute.