Section 2: Further trigonometric equations

Solutions to Exercise level 3

1. $\cos x = 2 + \sqrt{3}\sin x$

$\cos x - \sqrt{3}\sin x = 2$

$\cos x - \sqrt{3}\sin x = R\cos(x+\alpha)$

$= R\cos x\cos\alpha - R\sin x\sin\alpha$

$R\cos\alpha = 1$

$R\sin\alpha = \sqrt{3}$

$R^2 = 1 + 3 = 4 \quad \Rightarrow R = 2$

$\tan\alpha = \sqrt{3} \quad \Rightarrow \alpha = \frac{\pi}{3}$

$2\cos\left(x + \frac{\pi}{3}\right) = 2$

$2\cos\left(x + \frac{\pi}{3}\right) = 1$

$x + \frac{\pi}{3} = 2\pi, 4\pi, 6\pi, 8\pi$

$x = \frac{5\pi}{3}, \frac{11\pi}{3}, \frac{17\pi}{3}, \frac{23\pi}{3}$

The points of intersection are $\left(\frac{5\pi}{3}, \frac{1}{2}\right), \left(\frac{11\pi}{3}, \frac{1}{2}\right), \left(\frac{17\pi}{3}, \frac{1}{2}\right), \left(\frac{23\pi}{3}, \frac{1}{2}\right)$

2. Substituting $x = \frac{\pi}{4}$:

$2\sqrt{5}\sin x + 4\sqrt{5}\cos x = 2\sqrt{5}\sin\tfrac{\pi}{4} + 4\sqrt{5}\cos\tfrac{\pi}{4}$

$= 2\sqrt{5} \times \frac{1}{\sqrt{2}} + 4\sqrt{5} \times \frac{1}{\sqrt{2}}$

$= 2\sqrt{5} \times \frac{\sqrt{2}}{2} + 4\sqrt{5} \times \frac{\sqrt{2}}{2}$

$= \sqrt{10} + 2\sqrt{10}$

$= 3\sqrt{10}$

so $x = \frac{\pi}{4}$ is a root of the equation.

$2\sqrt{5}\sin x + 4\sqrt{5}\cos x = R\sin(x+\theta)$

$= R\sin x\cos\theta + R\cos x\sin\theta$

$R\cos\theta = 2\sqrt{5}$

$R\sin\theta = 4\sqrt{5}$

$R^2 = 4\times5+16\times5=100 \quad \Rightarrow R=10$

$\tan\theta = \dfrac{4\sqrt{5}}{2\sqrt{5}} = 2 \quad \Rightarrow \theta = \arctan 2$

$2\sqrt{5}\sin x + 4\sqrt{5}\cos x = 10\sin(x+\theta)$

Substituting $x = \dfrac{\pi}{4}$:

$3\sqrt{10} = 10\sin\left(\dfrac{\pi}{4}+\arctan 2\right)$

$\dfrac{3\sqrt{10}}{10} = \sin\left(\dfrac{\pi}{4}+\arctan 2\right)$

$\dfrac{3}{\sqrt{10}} = \sin\left(\dfrac{\pi}{4}+\arctan 2\right)$

Since $\dfrac{\pi}{4}+\arctan 2 > \dfrac{\pi}{2}$, before taking arcsin of both sides, need the principal value corresponding to $\dfrac{\pi}{4}+\arctan 2$. This is $\pi - \left(\dfrac{\pi}{4}+\arctan 2\right)$,

i.e. $\dfrac{3\pi}{4} - \arctan 2$

$\arcsin\left(\dfrac{3}{\sqrt{10}}\right) = \dfrac{3\pi}{4} - \arctan 2$

$\dfrac{3\pi}{4} = \arcsin\left(\dfrac{3}{\sqrt{10}}\right) + \arctan 2$

$\pi = \dfrac{4}{3}\left(\arcsin\left(\dfrac{3}{\sqrt{10}}\right) + \arctan 2\right)$

3. (i) $g(x) = 7\cos^2 x + \sin^2 x - 8\sin x\cos x$

$= 7\cos^2 x + 1 - \cos^2 x - 8\sin x\cos x$

$= 6\cos^2 x + 1 - 8\sin x\cos x$

$= 3(\cos 2x + 1) + 1 - 4\sin 2x$

$= 3\cos 2x - 4\sin 2x + 4$

$3\cos 2x - 4\sin 2x = b\cos(2x+\alpha)$

$= b\cos 2x\cos\alpha - b\sin 2x\sin\alpha$

$3 = b\cos\alpha$

$4 = b\sin\alpha$

$b^2 = 3^2 + 4^2 = 25 \quad \Rightarrow b = 5$

$\tan\alpha = \frac{4}{3}$

$g(x) = 4 + 5\cos(2x+\alpha)$

The greatest value of g(x) is 9 and the least value is -1.

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(ii) $4 + 5\cos(2x + \alpha) = 0$

$5\cos(2x + \alpha) = -4$

$\cos(2x + \alpha) = -\frac{4}{5}$

$2x + \alpha = 2.4981$

$x = 1.249 - \frac{1}{2}\alpha$

As α is acute, $\frac{1}{2}\alpha < \frac{\pi}{4}$ so this is the least positive value of x for which g(x) = 0.