Section 3: Geometric sequences and series

Solutions to Exercise level 3

1. (i) Scheme B is a geometric sequence, with $a = 20$ and $r = 1.12$

(ii) $u_n = 20(1.12)^{n-1}$

$$S_n = \frac{20(1.12^n - 1)}{1.12 - 1}$$
$$= \frac{500}{3}\left(1.12^n - 1\right)$$

(iii) $S_n = 3000 \Rightarrow \frac{500}{3}(1.12^n - 1) = 3000$

$\Rightarrow 1.12^n = 19$

$\Rightarrow n\log(1.12) = \log(19)$

$\Rightarrow n = \dfrac{\log(19)}{\log(1.12)} = 25.98$ (2 d.p.)

so Fred will have fully paid by the end of the 26th month.

(iv) $S_{25} = 500(1.12^{25} - 1) = 2666.68$ (2 d.p.)

so the payment in month 26 should be 3000 – 2666.68 = £333.32

2. (i) Consecutive terms of the series have neither a constant difference (except in the case where $x = 1$) nor a constant ratio – so the series is neither arithmetic or geometric.

(ii)
$$S_n = 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1}$$
$$xS_n = x + 2x^2 + 3x^3 + \ldots + (n-1)x^{n-1} + nx^n$$
$$(1-x)S_n = (1 + x + x^2 + x^3 + \ldots + x^{n-1}) - nx^n$$

The terms in the bracket are a geometric series with n terms

so $(1-x)S_n = \dfrac{1(x^n - 1)}{x - 1} - nx^n$

$$S_n = -\frac{x^n - 1}{(x-1)^2} + \frac{nx^n}{(x-1)}$$

(iii) For series A, put $x = 2, n = 4 \Rightarrow S_4 = -\dfrac{2^4 - 1}{(2-1)^2} + \dfrac{4(2^4)}{(2-1)} = 49$

and $1 + 4 + 12 + 32 = 49$

For series B, put $x = \frac{1}{2},\ n = 5 \Rightarrow S_5 = -\dfrac{(\frac{1}{2})^5 - 1}{(\frac{1}{2} - 1)^2} + \dfrac{5((\frac{1}{2})^5)}{(\frac{1}{2} - 1)} = \dfrac{57}{16}$

and $1 + 1 + \frac{3}{4} + \frac{1}{2} + \frac{5}{16} = \frac{57}{16}$

3. (i) At the end of the first 3 months, Marika's savings are

$$d_3 = M(1+r) + M(1+r)^2 + M(1+r)^3$$
$$= M(1+r)\left\{1 + (1+r) + (1+r)^2\right\}$$
$$= \frac{M(1+r)\left\{(1+r)^3 - 1\right\}}{(1+r)-1}$$
$$= \frac{M(1+r)\left\{(1+r)^3 - 1\right\}}{r} \quad \text{as required}$$

(ii) In a similar way, at the end of the first n months, Marika's savings are

$$d_n = M(1+r) + M(1+r)^2 + M(1+r)^3 + \dots\dots\dots\dots + M(1+r)^n$$
$$= M(1+r)\left\{1 + (1+r) + (1+r)^2 + \dots\dots\dots\dots + (1+r)^{n-1}\right\}$$
$$= \frac{M(1+r)\left\{(1+r)^n - 1\right\}}{r} \quad \text{as required}$$

(iii) Total to be saved = 15000, and $r = 0.005$

$$\Rightarrow 15000 = \frac{M(1.005)\left\{(1.005)^n - 1\right\}}{0.005}$$
$$\Rightarrow M = \frac{15000(0.005)}{(1.005)\left\{(1.005)^n - 1\right\}} = \frac{75}{(1.005)\left[(1.005)^n - 1\right]}$$

so if Marika saves for 3 years, $n = 36$,
and the monthly saving = £379.43
and if Marika saves for 4 years, $n = 48$,
and the monthly saving = £275.86

(iv)

Period of payment (months)	Monthly payment
36	£379.43
37	£368.23
38	£357.62
39	£347.55
40	£337.99
41	£328.90
42	£320.24
43	£311.99
44	£304.11
45	£296.58
46	£289.39
47	£282.50
48	£275.86

so by saving £300 per month, Marika can buy her car after 44 months.