Section 3: Geometric sequences and series

Solutions to Exercise level 2

1. (i) Common ratio $= \frac{4x-4}{5x+1}$ and $\frac{3x-5}{4x-4}$

$$\frac{4x-4}{5x+1} = \frac{3x-5}{4x-4}$$
$$(4x-4)^2 = (3x-5)(5x+1)$$
$$16x^2 - 32x + 16 = 15x^2 - 22x - 5$$
$$x^2 - 10x + 21 = 0$$
$$(x-3)(x-7) = 0$$
$$x = 3 \text{ or } x = 7$$

(ii) Common ratio $= \frac{4x-4}{5x+1}$

For $x = 3$, common ratio $= \frac{12-4}{15+1} = 0.5$

For $x = 7$, common ratio $= \frac{28-4}{35+1} = \frac{2}{3}$

2. (i) First term $a = 0.45$
Common ratio $r = 0.01$

(ii) $S_\infty = \frac{a}{1-r} = \frac{0.45}{1-0.01} = \frac{0.45}{0.99} = \frac{45}{99} = \frac{5}{11}$

3. $0.\dot{4}\dot{0}\dot{7} = 0.407 + 0.000407 + 0.000000407 + \ldots$
First term $a = 0.407$
Common ratio $r = 0.001$
$S_\infty = \frac{a}{1-r} = \frac{0.407}{1-0.001} = \frac{0.407}{0.999} = \frac{407}{999} = \frac{11}{27}$

4. Annual salary is a geometric sequence
First term $a = 18000$
Common ratio $r = 1.04$

(i) 10^{th} term $= ar^9 = 18000 \times 1.04^9 = £25619.61$ (to nearest penny)

(ii) $S_{10} = \frac{a(r^{10}-1)}{r-1} = \frac{18000(1.04^{10}-1)}{1.04-1} = £216109.93$ (to nearest penny)

5. Heights rebounded to after each bounce form a geometric sequence
After 1 bounce, it rebounds to $2\times0.8=1.6$ metres
First term $a=1.6$
Common ratio $r=0.8$

(i) nth term $=ar^{n-1}=1.6\times0.8^{n-1}$

$1.6\times0.8^{n-1}<0.1$

$0.8^{n-1}<0.0625$

$\log0.8^{n-1}<\log0.0625$

$(n-1)\log0.8^{n-1}<\log0.0625$

$n-1>\dfrac{\log0.0625}{\log0.8}$

$n-1>12.4$

$n>13.4$

It first rebounds to less than 10 cm after 14 bounces.

(ii) Total distance travelled $=2+2S_\infty$

$$=2+2\times\frac{a}{1-r}$$

$$=2+\frac{2\times1.6}{1-0.8}$$

$$=18 \text{ metres}$$

(iii) Total distance travelled after n bounces $=2+2S_n$

$$2+2\times\frac{1.6(1-0.8^n)}{1-0.8}>0.99\times18$$

$2+16(1-0.8^n)>17.82$

$1-0.8^n>0.98875$

$0.8^n<0.01125$

$n\log0.8<\log0.01125$

$$n>\frac{\log0.01125}{\log0.8}$$

$n>20.1$

After 21 bounces.

6. (i) $a=1,\ r=-\frac{1}{2}$

so $S_\infty=\dfrac{1}{1-(-\frac{1}{2})}=\dfrac{2}{3}$

(ii) Odd terms $A=1+\frac{1}{4}+\frac{1}{16}+\ldots$

so $a = 1, r = \frac{1}{4}$

$$S_\infty = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

(iii) Even terms $B = -\frac{1}{2} - \frac{1}{8} - \frac{1}{32} - \ldots$

so $a = -\frac{1}{2}, r = \frac{1}{4}$

$$S_\infty = \frac{-\frac{1}{2}}{1 - \frac{1}{4}}$$

$$= -\frac{1}{2} \times \frac{4}{3} = -\frac{2}{3}$$

(iv) For the full series, $S = A + B$

$\Rightarrow S_\infty = \frac{1}{3} - \frac{2}{3} = \frac{2}{3}$ as before