Section 2: Composite and inverse functions

Solutions to Exercise level 3

1. $ff(x) = (x+1)+1 = x+2$ $\quad gf(x) = \sqrt{(x+1)-1} = \sqrt{x}$

$(ff)^{-1}(x) = x-2$ $\quad (gf)^{-1}(x) = x^2$

$fg(x) = \sqrt{x-1}+1$ $\quad gg(x) = \sqrt{\sqrt{x-1}-1}$

$(fg)^{-1}(x) = (x-1)^2+1$ $\quad (fg)^{-1}(x) = (x^2+1)^2+1$

$ff(-1)=1$	$gf(1)=1$	$fg(1)=1$	$gg(5)=1$
$ff(0)=2$	$gf(4)=2$	$fg(2)=2$	$gg(26)=2$
$ff(1)=3$	$gf(9)=3$	$fg(5)=3$	$gg(101)=3$
$ff(2)=4$	$gf(16)=4$	$fg(10)=4$	$gg(290)=4$
$ff(3)=5$	$gf(25)=5$	$fg(17)=5$	$gg(677)=5$

2. (i) $h^2(x) = \dfrac{1}{1/x} = x$

(ii) $gj(x) = g(4x) = \sqrt{4x} = 2\sqrt{x}$

(iii) $jf(x) = j(4^x) = 4\times 4^x = 4^{x+1}$

(iv) $gf(x) = g(4^x) = \sqrt{4^x} = 2^x$

(v) $jgj(x) = jg(4x) = j\left(\sqrt{4x}\right) = j\left(2\sqrt{x}\right) = 4\times 2\sqrt{x} = 8\sqrt{x}$

or alternatively $gjjj(x) = g(64x) = 8\sqrt{x}$

3. $f(x) = ax^2+b$

$f^{-1}(1) = 1 \Rightarrow f(1) = 1 \Rightarrow a+b = 1$

$f^{-1}(2) = 2 \Rightarrow f(2) = 2 \Rightarrow 4a+b = 2$

Subtracting: $3a = 1 \Rightarrow a = \frac{1}{3}, b = \frac{2}{3}$

$f^{-1}(3) = k \quad \Rightarrow f(k) = 3$

$\Rightarrow \frac{1}{3}k^2 + \frac{2}{3} = 3$

$\Rightarrow k^2 = 7$

So $f^{-1}(3) = \sqrt{7}$

4. (i) $f(x) = x^2 + 4x + 3 = (x+2)^2 - 1$

So the minimum point is (-2, 1), and hence $k = -2$

(ii) $y = (x+2)^2 - 1$ for $x \leq -2$

For inverse function, $x = (y+2)^2 - 1$

$x + 1 = (y+2)^2$

$-\sqrt{x+1} = y + 2$

$y = -\sqrt{x+1} - 2$

As $x \leq -2$, take negative square root

Domain of inverse function is range of f(x) which is $f(x) \geq -1$

So inverse is $f^{-1}(x) = -\sqrt{x+1} - 2, \quad x \geq -1$

(iii) They are reflections of each other in the line $y = x$.

(iv) If the graphs do not meet, then neither of them meet the line $y = x$.

If they do meet, $x^2 + 4x + 3 = x$

$\Rightarrow x^2 + 3x + 3 = 0$

Discriminant of quadratic equation is $9 - 4 \times 1 \times 3 = 9 - 12 = -3$

So the equation has no real roots, and therefore the graphs do not meet.