Section 1: Methods of proof

Solutions to Exercise level 2

1. (i) a and b are both positive $\Rightarrow |a+b| = |a| + |b|$

   (ii) a and b are both negative $\Rightarrow |a+b| = |a| + |b|$

   (iii) a is positive, b is negative and $|a| > |b|$ $\Rightarrow |a+b| < |a|$
   $\Rightarrow |a+b| < |a| + |b|$

   (iv) a is positive, b is negative and $|a| < |b| \Rightarrow |a+b| < |b|$
   $\Rightarrow |a+b| < |a| + |b|$

   (v) a is negative, b is positive and $|a| > |b| \Rightarrow |a+b| < |a|$
   $\Rightarrow |a+b| < |a| + |b|$

   (vi) a is negative, b is positive and $|a| < |b| \Rightarrow |a+b| < |b|$
   $\Rightarrow |a+b| < |a| + |b|$

   so $|a+b| \le |a| + |b|$ in all cases.

2. (i) $10^{3n} + 1 = (10^n + 1)(10^{2n} - 10^n + 1)$

   (ii) $10^n + 1 > 1$ for all integer values of n greater than 0, since $10^n > 0$ for all integer values greater than 0.
   $10^{2n} - 10^n + 1 = 10^n(10^n - 1) + 1$
   Since $10^n \ge 10$ for all integer values of n greater than 0, then $10^n(10^n - 1) + 1 > 1$ for all integer values of n greater than 0.
   Therefore $10^{3n} + 1$ may be written as the product of two factors, which are both integers greater than 1, so it is not a prime number.

3. (i) Assume that there is a rational number $\frac{p}{q}$ in its lowest terms, with p, $q \ne 0$, which satisfies the equation.

   Then $x^3 + x + 1 = 0 \Rightarrow \frac{p^3}{q^3} + \frac{p}{q} + 1 = 0$
   $\Rightarrow p^3 + pq^2 + q^3 = 0$

   (ii) (a) If p and q are both odd:
   p^3 is odd, pq^2 is odd and q^3 is odd,
   so $p^3 + pq^2 + q^3$ is the sum of three odd numbers and is therefore odd.

   (b) If p is even and q is odd:
   p^3 is even, pq^2 is even and q^3 is odd,

so $p^3 + pq^2 + q^3$ is the sum of two even and one odd number and is therefore odd.

(c) If p is odd and q is even:
p^3 is odd, pq^2 is even and q^3 is even,
so $p^3 + pq^2 + q^3$ is the sum of two even and one odd number and is therefore odd.

So in each of these cases the left-hand side of the equation is odd and therefore cannot be zero.

(iii) If p and q are both even then 2 is a common factor of both p and q, so $\frac{p}{q}$ is not a fraction in its lowest terms. This contradicts the assumption in (i).

4. Assume that there is a rational number $\frac{p}{q}$ in its lowest terms, with $p, q \neq 0$, which satisfies the equation.

Then $x^5 + x^4 + x^3 + x^2 + 1 = 0 \Rightarrow \frac{p^5}{q^5} + \frac{p^4}{q^4} + \frac{p^3}{q^3} + \frac{p^2}{q^2} + 1 = 0$

$\Rightarrow p^5 + p^4q + p^3q^2 + p^2q^3 + q^5 = 0$

If p and q are both odd:
p^5 is odd, p^4q is odd, p^3q^2 is odd, p^2q^3 is odd and q^5 is odd,
so $p^5 + p^4q + p^3q^2 + p^2q^3 + q^5$ is the sum of five odd numbers and is therefore odd.

If p is even and q is odd:
p^5 is even, p^4q is even, p^3q^2 is even, p^2q^3 is even and q^5 is odd,
so $p^5 + p^4q + p^3q^2 + p^2q^3 + q^5$ is the sum of four even and one odd number and is therefore odd.

If p is odd and q is even:
p^5 is odd, p^4q is even, p^3q^2 is even, p^2q^3 is even and q^5 is even,
so $p^5 + p^4q + p^3q^2 + p^2q^3 + q^5$ is the sum of four even and one odd number and is therefore odd.

So in each of these cases the left-hand side of the equation is odd and therefore cannot be zero.

If p and q are both even then 2 is a common factor of both p and q, so $\frac{p}{q}$ is not a fraction in its lowest terms. This contradicts the original assumption.

Edexcel A level Maths Proof 1 Exercise solutions

5. (i) $n(n-1)(n+1)(n^2+n+1)(n^2-n+1)$

$= n(n-1)(n+1)((n^2+1)+n)((n^2+1)-n)$

$= n(n^2-1)((n^2+1)^2-n^2)$

$= n(n^2-1)(n^4+2n^2+1-n^2)$

$= n(n^2-1)(n^4+n^2+1)$

$= n(n^6-n^4+n^4-n^2+n^2-1)$

$= n^7-n$

(ii) If $n = 7x$ where x is a positive integer then n is clearly divisible by 7.

If $n = 7x + 1$ where x is a positive integer
then $n - 1 = 7x + 1 - 1 = 7x$ which is divisible by 7.

If $n = 7x + 2$ where x is a positive integer
then $n^2+n+1 = (7x+2)^2+7x+2+1$

$= 49x^2+35x+7$

$= 7(7x^2+5x+1)$

which is divisible by 7.

If $n = 7x + 3$ where x is a positive integer
then $n^2-n+1 = (7x+3)^2-7x-3+1$

$= 49x^2+35x+7$

$= 7(7x^2+5x+1)$

which is divisible by 7.

If $n = 7x + 4$ where x is a positive integer
then $n^2+n+1 = (7x+4)^2+7x+4+1$

$= 49x^2+63x+21$

$= 7(7x^2+9x+3)$

which is divisible by 7.

If $n = 7x + 5$ where x is a positive integer
then $n^2-n+1 = (7x+5)^2-7x-5+1$

$= 49x^2+63x+21$

$= 7(7x^2+9x+3)$

which is divisible by 7.

If $n = 7x + 6$ where x is a positive integer
then $n+1 = 7x+6+1 = 7x+7 = 7(x+1)$
which is divisible by 7.