Section 1: Methods of proof

Solutions to Exercise level 1

1. (i) True

Using direct proof:

a is a factor of $b \Rightarrow ma = b$ for some integer m

a is a factor of $c \Rightarrow na = c$ for some integer n

$b + c = ma + na = (m+n)a$

Since $m + n$ is an integer, a is a factor of $b + c$.

(ii) True

Using proof by exhaustion:

2 is not a factor of 103

3 is not a factor of 103

5 is not a factor of 103

7 is not a factor of 103

11 is not a factor of 103

Since $11 > \sqrt{103}$, there can be no other prime factors of 103, and therefore 103 has no factors other than itself and 1.

103 is therefore a prime number.

(iii) False

$n = 4$ is a counter-example, since for $n = 4$, $n^2 + n + 1 = 21$, which is not prime.

(iv) True

Using direct proof:

$(a-b)^2 \geq 0$ for all real numbers a and b

$\Rightarrow a^2 - 2ab + b^2 \geq 0$

$\Rightarrow a^2 + b^2 \geq 2ab$

(v) True

Using proof by contradiction:

Assume that the cube root of 2 is rational, i.e. $\sqrt[3]{2} = \frac{a}{b}$ for integers a and b, where a and b have no common factor.

$$\Rightarrow 2 = \frac{a^3}{b^3}$$

$$\Rightarrow a^3 = 2b^3$$

Since $2b^3$ is even, a^3 is even, so a is even and can therefore be written as $a = 2p$.

$$a^3 = 2b^3 \Rightarrow (2p)^3 = 2b^3$$
$$\Rightarrow 8p^3 = 2b^3$$
$$\Rightarrow 4p^3 = b^3$$

Since $4p^3$ is even, b^3 is even, so b is even.
Therefore a and b are both even, so they have a common factor of 2, contradicting the original assumption.
Therefore the cube root of 2 is irrational.

2. (i) a is a rational number $\Rightarrow a = \frac{m}{n}$ for some integers m and n.

b is a rational number $\Rightarrow b = \frac{p}{q}$ for some integers p and q.

$$a + b = \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}$$

Since $mq + np$ and nq are both integers, $a + b$ is a rational number.

(ii) For example, $a = \sqrt{2}$, $b = 1 - \sqrt{2}$, so $a + b = 1$. So $a + b$ is a rational number but a and b are not.

(iii) Assume that a is rational, b is irrational and $a + b$ is rational.

Since a is rational, it may be written as $\frac{m}{n}$, where m and n are integers with no common factor.

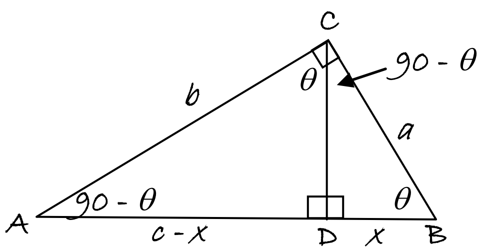
Since $a + b$ is rational, it may be written as $\frac{p}{q}$, where p and q are integers with no common factor.

$$\frac{m}{n} + b = \frac{p}{q}$$

$$b = \frac{p}{q} - \frac{m}{n} = \frac{np - mq}{nq}$$

Since $np - mq$ and nq are both integers, b is rational, contradicting the original assumption.
So if a is rational and b is irrational, $a + b$ is irrational.

3.



(i) Let angle ACD be θ. Then angle CAD $= 90° - \theta$.
Also angle BCD $= 90° - \theta$, so angle CBD $= \theta$.
Triangles ADC and ACD both have angles of $90°$, θ and $90° - \theta$

$$\frac{c-x}{b} = \frac{b}{c}$$

(ii) Triangles BDC and BCA both have angles of $90°$, θ and $90° - \theta$

$$\frac{x}{a} = \frac{a}{c}$$

(iii) From (ii), $x = \frac{a^2}{c}$

Substituting into the relationship from (i):

$$c\left(c - \frac{a^2}{c}\right) = b^2$$

$$\Rightarrow c^2 - a^2 = b^2$$

$$\Rightarrow a^2 + b^2 = c^2$$

This is a proof of Pythagoras' theorem.

4. (i) $x^2 - y^2 = 1$

$(x+y)(x-y) = 1$

(ii) Assume that a solution (x, y) exists where x and y are positive integers.

(iii) If a solution exists, then
either $x + y = 1$ and $x - y = 1$
or $x + y = -1$ and $x - y = -1$

(iv) In the first case: $x + y = 1$

$x - y = 1$

$2x = 2$

$x = 1,\ y = 0$

In the second case: $x + y = -1$

$x - y = -1$

$2x = -2$

$x = -1,\ y = 0$

This contradicts the original assumption that both x and y are positive integers. So there is no solution where x and y are positive integers.

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5. $x^2 - y^2 = 10$

$(x+y)(x-y) = 10$

Assume that a solution (x, y) exists where x and y are positive integers.
If a solution exists, then

either $x + y = 2$ and $x - y = 5$
or $x + y = -2$ and $x - y = -5$
or $x + y = 5$ and $x - y = 2$
or $x + y = -5$ and $x - y = -2$

In the first case:
$$x + y = 2$$
$$\underline{x - y = 5}$$
$$2x = 7$$
$$x = 3.5,\ y = -1.5$$

In the second case:
$$x + y = -2$$
$$\underline{x - y = -5}$$
$$2x = -7$$
$$x = -3.5,\ y = 1.5$$

In the third case:
$$x + y = 5$$
$$\underline{x - y = 2}$$
$$2x = 7$$
$$x = 3.5,\ y = 1.5$$

In the fourth case:
$$x + y = -5$$
$$\underline{x - y = -2}$$
$$2x = -7$$
$$x = -3.5,\ y = -1.5$$

This contradicts the original assumption that both x and y are positive integers. So there is no solution where x and y are positive integers.