Edexcel A level Maths Projectiles

Section 2: General equations

Notes and Examples

These notes contain subsections on
- The equation of the trajectory of a projectile
- Further examples

The equation of the trajectory of a projectile

If you eliminate the t variable from the equations for vertical displacement (y) and horizontal displacement (x), you get a Cartesian equation, which gives you the projectile's path through the air.

The path equation is very useful. It gives you a way of identifying each point on the trajectory easily.

Example 1
A particle is projected from a point 1 m above the ground, with initial velocity of 10 ms^{-1} at an angle θ above the horizontal, where $\cos\theta = 0.8$ and $\sin\theta = 0.6$. The origin is taken to be the point on the ground directly below the point of projection.
(i) Find equations for the horizontal and vertical positions, x m and y m, in terms of t.
(ii) Find the Cartesian equation of the trajectory of the particle.
(iii) Find the value of y when x is 2.
(iv) Find the possible values of x when y is 2.

Solution
(i) The initial horizontal speed of the particle is $10\cos\theta = 10\times 0.8 = 8$
Horizontally the speed is constant
so $x = 8t$

The initial vertical speed of the particle is $10\sin\theta = 10\times 0.6 = 6$
The vertical displacement is given by $s = ut + \frac{1}{2}at^2$

$$= 6t + \tfrac{1}{2}\times -10t^2$$
$$= 6t - 5t^2$$

The initial height is 1 m
so $y = 1 + 6t - 5t^2$

(ii) $x = 8t \Rightarrow t = \dfrac{x}{8}$
Substituting into $y = 1 + 6t - 5t^2$:

$$y = 1 + \frac{6x}{8} - 5\left(\frac{x}{8}\right)^2$$
$$y = 1 + \frac{3}{4}x - \frac{5}{64}x^2$$

(iii) When $x = 2$, $y = 1 + \frac{3}{4} \times 2 - \frac{5}{64} \times 2^2$
$$= 1 + \frac{3}{2} - \frac{5}{16}$$
$$= 2.1875$$

(iii) When $y = 2$, $2 = 1 + \frac{3}{4}x - \frac{5}{64}x^2$
$$\frac{5}{64}x^2 - \frac{3}{4}x + 1 = 0$$
$$5x^2 - 48x + 64 = 0$$
$$(x-8)(5x-8) = 0$$
$$x = 8 \quad \text{or} \quad x = 1.6$$

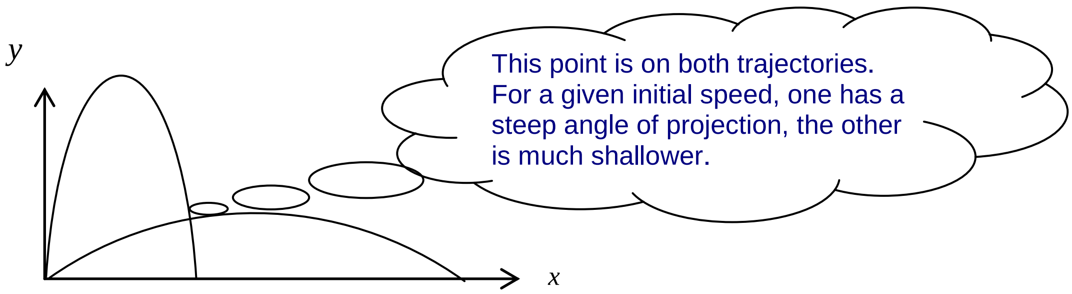
In the same way, it is possible to find the general equation for the path of a projectile in terms of its initial velocity V and angle of projection α:

$$y = x\tan\alpha - \frac{gx^2}{2V^2}\left(1 + \tan^2\alpha\right)$$

Notice that this equation is quadratic in both α and x. This means that for each possible point on a projectile's path, for every initial velocity there are two angles which will result in the projectile passing through the point, except at maximum range.
(for a quadratic equation, for all y values, except the maximum (or minimum) point, there are two possible x values – think about the graph of a quadratic)

You should **NOT** attempt to learn or quote this equation. If you are asked to find the equation of a trajectory, you will be asked to eliminate t from the equations for x and y, as in Example 1 above.

There are usually two ways of "hitting" a point with a given initial speed. One path has a low trajectory and a short time of flight, whereas the other has a much higher trajectory and a longer time of flight.



Edexcel A level Maths Projectiles 2 Notes & Examples

In all this work you have assumed ideal conditions, with, in particular, no air resistance. In reality this can be quite important as it can have various effects on the particle, as well as just slowing it down e.g. spinning and turning.

Further examples

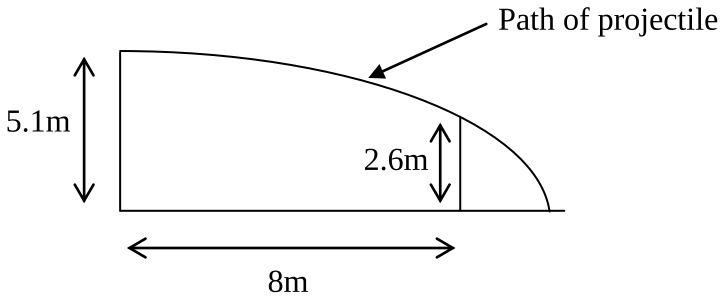
Example 2

A boy throws a ball horizontally from a point 5.1 m above the horizontal ground.

(i) What is the minimum speed at which the ball must be thrown to clear a fence 2.6 m high at a horizontal distance of 8 m from the point of projection?

(ii) Find the distance beyond the fence at which the ball strikes the ground if it is projected at this speed?

Solution



(i) In the time the ball travels 8 m horizontally, it must fall a maximum distance of 5.1 – 2.6 = 2.5 m if it is to clear the fence.

Considering the vertical motion:
The time to fall 2.5 m:

$u = 0$

$a = 9.8\text{ ms}^{-2}$

$s = 2.5\text{ m}$

$t = ?$

$s = ut + \frac{1}{2}at^2$

$2.5 = 4.9t^2$

$t = 0.714\text{ seconds (3 s.f.)}$

So the ball travels 8 m horizontally in 0.714 s,

so its initial horizontal speed is $\frac{8}{0.714} = 11.2\text{ ms}^{-1}$ (3 s.f.), or greater.

(ii) The time for the ball to fall to ground level is found as above, except that now s = 5.1 m, instead of 2.5 m.

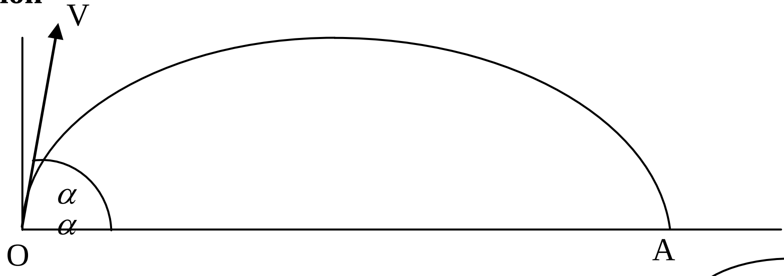
So $5.1 = 4.9t^2 \Rightarrow t = \sqrt{\frac{5.1}{4.9}} = 1.02\text{ s}$ (3 s.f.)

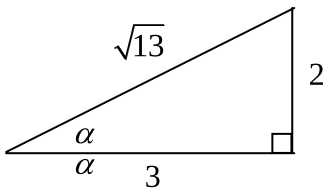
In 1.02 seconds the ball will travel $1.02 \times 11.2 = 11.42\text{m}$ (4 s.f.) horizontally, so it will land 11.42 - 8 = 3.42 m (3 s.f.) beyond the fence.

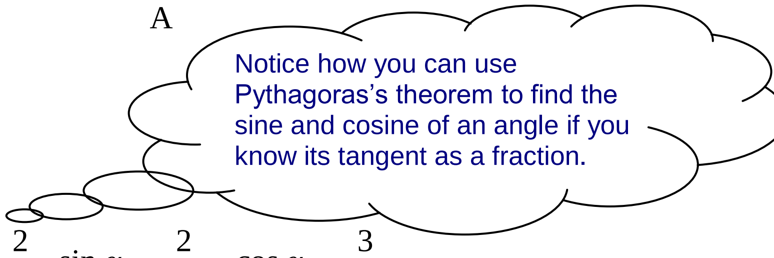
Example 3

A projectile is fired with an initial speed V from a point O on a horizontal plane, the angle of elevation being $\tan^{-1}\left(\frac{2}{3}\right)$. The particle returns to the plane at a point A. Find the distance OA, in terms of V. Show that the same point could have been reached by firing the particle at the same speed from O, but at an angle of elevation $\tan^{-1}\left(\frac{3}{2}\right)$. Find, in terms of V, the difference in the times of flights of the two trajectories.

Solution







$$\tan\alpha = \frac{2}{3},\quad \sin\alpha = \frac{2}{\sqrt{13}},\quad \cos\alpha = \frac{3}{\sqrt{13}}$$

To find the time of flight, since the particle lands on the same level as it started, we can find the time to maximum height and then double it:

Vertically:

$u = V\sin\alpha = \frac{2}{\sqrt{13}}V$

$v = 0$ (at max height)

$a = -9.8$

$t = ?$

$$v = u + at$$
$$0 = \frac{2V}{\sqrt{13}} - 9.8t$$
$$t = \frac{2V}{9.8\sqrt{13}}$$

The time of flight is double this,

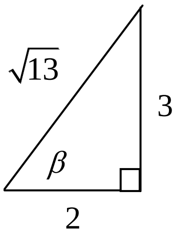
so the time of flight is $t = \frac{4V}{9.8\sqrt{13}} = 0.113V$ seconds (3 s.f)

The horizontal range is the value of x when t takes the above value.

$$\begin{aligned} x &= Vt\cos\alpha \\ &= V \times \frac{4V}{9.8\sqrt{13}} \times \frac{3}{\sqrt{13}} \\ &= \frac{12V^2}{9.8\times 13} \\ &= 0.094V^2 \text{ (3 d.p.)} \end{aligned}$$

If the particle is projected at the same speed, V, but at angle β, such that $\tan\beta = \dfrac{3}{2}$,

then:



$$\tan\beta = \frac{3}{2},\ \sin\beta = \frac{3}{\sqrt{13}},\ \cos\beta = \frac{2}{\sqrt{13}}$$

To find the time of flight, find the time to maximum height and then double it, as before:

Vertically:

$u = V\sin\alpha = \dfrac{3}{\sqrt{13}}V$

$v = 0$ (at max height)

$a = -9.8$

$t = ?$

$$v = u + at$$

$$0 = \frac{3V}{\sqrt{13}} - 9.8t$$

$$t = \frac{3V}{9.8\sqrt{13}}$$

The time of flight is double this,

So the time of flight is $\dfrac{6V}{9.8\sqrt{13}} = 0.170V$ seconds (3 s.f).

The horizontal range is the value of x when t takes the above value.

$$\begin{aligned} x &= Vt\cos\alpha \\ &= V \times \frac{6V}{9.8\sqrt{13}} \times \frac{2}{\sqrt{13}} \\ &= \frac{12V^2}{9.8\times 13} \\ &= 0.094V^2 \text{ (3 d.p.)} \end{aligned}$$

So the ranges are the same.

The difference in the times of flight is $\dfrac{6V}{9.8\sqrt{13}} - \dfrac{4V}{9.8\sqrt{13}} = \dfrac{2V}{9.8\sqrt{13}} = 0.0566V$ seconds

(3 s.f.)