Section 1: Motion in two and three dimensions

Notes and examples

These notes contain subsections on
- Constant acceleration in two dimensions
- Variable acceleration in two dimensions
- The equation of a path

Constant acceleration in two and three dimensions

In AS / Year 1 Mathematics you used the constant acceleration formulae. These can also be used in two dimensions, using vectors for displacement, velocity and acceleration.

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$\mathbf{r} = \tfrac{1}{2}(\mathbf{u} + \mathbf{v})t$$
$$\mathbf{r} = \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2$$
$$\mathbf{r} = \mathbf{v}t - \tfrac{1}{2}\mathbf{a}t^2$$

The fifth constant acceleration formula $v^2 = u^2 + 2as$ has a vector form as well but this is beyond the scope of this course.

Example 1
A particle is initially at the origin with velocity $(3\mathbf{i} - 2\mathbf{j})$ ms^{-1} has constant acceleration of $(\mathbf{i} + 3\mathbf{j})$ ms^{-2}. Find its velocity and position vector after 4 seconds.

Solution

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$= \begin{pmatrix} 3 \\ -2 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

$$\mathbf{r} = \mathbf{u}t + \tfrac{1}{2}\mathbf{a}t^2$$
$$= 4\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \tfrac{1}{2} \times 4^2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 20 \\ 16 \end{pmatrix}$$

Variable acceleration in two dimensions

In AS / Year 1 Mathematics you looked at how calculus can be used to work with variable acceleration in one dimension. This can be extended to two or three dimensions, using vectors. If vectors are given in component form, with one or more components which are a function of time, then you can differentiate or integrate each component as required.

The key to this work is to remember when to differentiate and when to integrate.
If you are given the position vector, you differentiate once to find the velocity vector (each component, of course) and differentiate again to find the acceleration.

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \qquad \mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

It tends to be slightly harder to find the velocity vector or position vector given the acceleration vector. You integrate in these cases, but remember the constants of integration, which will be vectors. These must be worked out at each stage, before you go onto the next.

$$\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t \qquad \mathbf{r} = \int \mathbf{v}\,\mathrm{d}t$$

Example 2
The force acting on a particle of mass 5 kg is given as ($20\mathbf{i} + 15t\mathbf{j}$) N. If its initial velocity is ($3\mathbf{i} + 25\mathbf{j}$) ms^{-1}, find its position after 3 seconds, given that its position after 1 second was ($8\mathbf{i} + 7\mathbf{j}$) m.

Solution
$\mathbf{F} = m\mathbf{a}$ (Newton's second law), so its acceleration is $\frac{1}{5}(20\mathbf{i}+15t\mathbf{j}) = 4\mathbf{i} + 3t\mathbf{j}$

Integrating each component gives its velocity as

$$(4t+c)\mathbf{i} + \left(\frac{3}{2}t^2 + d\right)\mathbf{j}$$

You need to find the values of the integration constants, c and d.
Its velocity at $t = 0$ was $3\mathbf{i} + 25\mathbf{j}$. This means $c = 3$ and $d = 25$.
Therefore the velocity is $(4t+3)\mathbf{i} + \left(\frac{3}{2}t^2 + 25\right)\mathbf{j}$

Integrating each component gives its position vector as

$$\mathbf{r} = (2t^2 + 3t + e)\mathbf{i} + \left(\frac{1}{2}t^3 + 25t + f\right)\mathbf{j}$$

You need to find the values of the integration constants e and f.
The displacement after 1 second was $8\mathbf{i} + 7\mathbf{j}$. This means $e = 3$ and $f = -18.5$

Therefore $\mathbf{r} = \left(2t^2 + 3t + 3\right)\mathbf{i} + \left(\frac{1}{2}t^3 + 25t - 18.5\right)\mathbf{j}$

So, when $t = 3$, the position is $(30\mathbf{i} + 70\mathbf{j})$ m.

The equation of a path

If the position of a particle is given in vector form in terms of time, it may be possible to eliminate t to give a Cartesian equation for the path of the particle (i.e. an equation connecting x and y).

Example 3
The position of a particle is given by $\mathbf{r} = (t^2 - 2)\mathbf{i} + (t + 1)\mathbf{j}$.
Find a Cartesian equation for the path of the particle.

Solution

$$y = t + 1 \quad \Rightarrow t = y - 1$$

$$\begin{aligned} x = t^2 - 2 &= (y-1)^2 - 2 \\ &= y^2 - 2y + 1 - 2 \\ &= y^2 - 2y - 1 \end{aligned}$$

A Cartesian equation for the path is $x = y^2 - 2y - 1$.